A Variant Triplet Model and Integral Charge

Yasuhiro Katayama, Isao Umemura and Eiji Yamada

Research Institute for Fundamental Physics
Kyoto University, Kyoto

(Received June 30, 1965)

A variant form of Sakata model for elementary particles is proposed. The fundamental triplet is assumed to obey para-Fermi statistics with capacity 3, and to be characterized by the three quantum numbers, the third component of isospin, the strangeness and the triplet number. It is not an eigenstate of electric and hyper charges, while it has definite expectation values for both charges, (2/3, -1/3, -1/3) and (1/3, 1/3, -2/3) respectively.

The triplet state can be treated as mixture of three sets of sub-triplets which are ordinary fermions with an appropriate description. It is possible to assign them integral eigenvalues for electric and hyper charges.

Under the restriction that the physically realizable states should be eigenstates of both charges, it is shown that most of lowlying states are the ones which are actually observed. Although the isolated existence of the triplet state is ruled out, the group theoretical arguments for the realizable states, such as the ratio of magnetic moments of baryons in $U(6)$ symmetry, are not affected.

§1. Introduction

The proposal of meson theory by Yukawa, which was made thirty years ago, was not a mere addition of one more member to a family of elementary particles, but was an attempt to have a unified understanding of various nuclear phenomena by means of the heavy quantum. The meson field was introduced in analogy with electromagnetic field, but it was assumed to carry an electric charge besides its heavy mass which was necessitated by the short range character of nuclear force. The charge was necessary to explain $\beta$-decay of nucleus and cosmic ray phenomena with the same meson. It seems rather fortunate that we were not under the influence of powerful mathematical formalism which allowed the existence of fractional charge at that time as we are at present, since purely formal consideration based on the property of nuclear force can not rule out the possibility of fractional charge.

Let us assume for the moment that we have not yet recognized the existence of pi-meson and that our experimental knowledge is restricted to narrow neighbourhood of nucleon. Then we may have introduced the concept of fractional charge through the measurement of magnetic moment of nucleon and electron scattering by nucleon, because it
A Variant Triplet Model and Integral Charge

seems rather natural to introduce fractional charge than to think of probability distribution of pi-meson cloud. A similar consideration can be applied to quark or Ace of present day,\textsuperscript{9} too. The fact that only some special collection of quarks is observed while the individual one can not be found is quite analogous to the case of pi-meson around nucleon where only the cloud of meson, and not the meson itself, is observed.

With such a consideration, we may conjecture that the appearance of fractional charge in quark model comes from the over determination of a physical quantity, electric charge, by the neglection of so far unknown sub-level. In fact, if we assume that the fundamental triplet\textsuperscript{9} is not a pure state but a mixed state with regard to the charge, it is not strange at all to get fractional value for the expectation value of the charge.

The possibility that the fundamental triplet is composed of the material belonging to the sub-level has been suggested by the consideration on the origin of \(SU(6)\) symmetry\textsuperscript{6} from substantial point of view. Greenberg\textsuperscript{6} has shown that the assignment of 56-dimensional representation of \(SU(6)\) to baryons and their excited states can be reasonably understood with quark model when we assume that the quark obeys para-Fermi (Okayama’s) statistics of capacity 3.\textsuperscript{6} Such para-quark can be expressed in the following way with the representation given by Green,

\[
q_\lambda = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} t_{\lambda \alpha}, \quad \lambda = 1, 2, 3.
\]  

(1)

Three sets of triplet fields, \(t_{\lambda \alpha} (\alpha = 1, 2, 3)\) commute each other, and satisfy anticommutation relations within the same set. If we introduce dual structure for vacuum, we have another type of representation for the triplet field,

\[
q_\lambda = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} \omega_\alpha f_{\lambda \alpha}, \quad \omega_\alpha \omega_\beta = \delta_{\alpha \beta} + i \epsilon_{\alpha \beta \gamma} \omega_\gamma.
\]  

(2)

Here, \(f_{\lambda \alpha}\)’s satisfy anticommutation relations among themselves, and \(\omega_\alpha\)’s are Pauli matrices operating on vacuum. Thus the fundamental triplet \(q_\lambda\) is interpreted to be composed of three sets of sub-triplets \(f_{\lambda \alpha}\), and the fundamental triplet state is not a pure but a mixed state with regard to some of physical quantities defined by the sub-triplets.

We assume that electric charge of \(f_{\lambda \alpha}\)’s is given by \((Q_\alpha + 1, Q_\alpha, Q_\alpha)\) satisfying the condition

\[
Q_1 + Q_2 + Q_3 = -1.
\]  

(3)

---

\textsuperscript{9} In this paper we call the fundamental field triplet to distinguish it from quark.
The assignment of baryons and their decuplet excited states to 56-dimensional representation of $SU(6)$ gives correct values for the charge. The same result is obtained for $U(3)$ symmetry too, if the assumption is made that baryons and the excited states are symmetric three body composite systems of the triplet.

There are two possibilities under the condition (3). One of them is to take $Q_\alpha = -1/3$. This case is the original quark model, where the triplet $q_3$ is an eigenstate of electric charge with the value $(2/3, -1/3, -1/3)$. The other possibility is given by the case where at least one of $Q_\alpha$ is different from others. Then the triplet becomes mixed state of electric charge. The expectation value of the charge is again $(2/3, -1/3, -1/3)$. The assignment of the integral charges to sub-triplets necessarily realizes the second possibility. The simplest assignment $(1, 0, 0), (1, 0, 0)$ and $(0, -1, -1)$ is same as the values for three sets of triplets discussed by Nambu.7

Similar argument can be made on hypercharge. If the fundamental triplet is an eigenstate of the third component of isospin with the eigenvalue $(1/2, -1/2, 0)$, the sub-triplets have the hypercharge $(Y_\alpha, Y_\alpha, Y_\alpha, -1)$ due to Nakano-Nishijima-Gell-Mann formula\(^8\) for them, $Q_\alpha = (Y_\alpha - 1)/2$. Here, $Y_\alpha$'s satisfy the condition

$$Y_1 + Y_2 + Y_3 = 1.$$  \hspace{1cm} (4)

In the first possibility where $Y_\alpha = 1/3$, the fundamental triplet is a pure state with the eigenvalue $(1/3, 1/3, -2/3)$, while in the other case it is a mixture with the expectation value same as the above eigenvalue. When the triplet is assumed to be an eigenstate of the third component of isospin, it is pure or mixed state for both of electric and hyper charges simultaneously. The hypercharges of sub-triplets are $(1, 1, 0), (1, 1, 0)$ and $(-1, -1, -2)$ in the above assignment of electric charge. The baryon number is defined for the sub-triplets as a quantity which takes the values $1, 1$ and $-1$ for $\alpha = 1, 2$ and $3$ respectively. With this assignment, the triplet also becomes an eigenstate of strangeness with eigenvalue $(0, 0, -1)$. There is another quantum number which has unit eigenvalue for each triplet state. We shall call it triplet number, since it cannot be identified with baryon number.

Although our model is quite similar as the original Sakata model,\(^9\) it has an important difference from this model. The fundamental triplet in our model is not an eigenstate of electric charge and has no particle-like character in ordinary sense. Since we can observe only eigenstates of electric charge, the realizable states are restricted to rather a small class of whole states. For instance, only $1^-, 35^-$ and $56$-dimensional representations and their direct products can be realized in the case of
$U(6)$ symmetry. When the symmetry of the system is $U(3) \times SU(2)$, the realizable states are obtained by the decomposition of the above representations of $U(6)$. Those states are expressed as the composite systems of several fundamental triplets and their charge conjugates, and they obey the ordinary Bose or Fermi statistics for even or odd triplet number respectively. The group theoretical conclusions on the realizable states are not different from the ordinary cases.

§2. Para-fermion triplets

We consider the triplet field with spin $1/2$, $\psi_\lambda (\lambda = 1, 2, 3)$ which obeys a generalized Fermi statistics—para-fermi statistics—with capacity 3. The introduction of para-statistics is needed, as suggested by Greenberg, to form the three body composite system of the $S$-state triplets in symmetrical way.

The field quantity $\psi_\lambda$ is expanded as

$$\psi_\lambda = \sum_{k, \alpha=1}^{3} \left[ \frac{2M}{E_k} \left( u_\alpha^k q_\alpha(k) e^{iks} + \eta_\alpha^k r_\alpha^k(k) e^{-iks} \right) \right].$$  \hspace{1cm} (5)

The variables $q_\alpha(k)$ and $r_\lambda^k(k)$ appearing in the expansion satisfy the following commutation relations,

$$[[q_\lambda, q_\mu], q_\sigma] = 0,$$

$$[[q_\lambda, q_\mu^*], q_\sigma] = \frac{2}{3} \delta_\mu^* q_\lambda, \hspace{1cm} \text{(6)}$$

and

$$[[[q_\sigma, q_\lambda], q_\mu], q_\nu] = 0,$$

$$[[[q_\sigma, q_\lambda^*], q_\mu^*], q_\nu] = 2\delta_\lambda^\mu [q_\alpha, q_\sigma] + \frac{4}{3} \delta_\lambda^\mu [q_\alpha, q_\nu] + \frac{2}{3} \delta_\nu^\alpha [q_\lambda, q_\mu] +$$

$$+ \frac{4}{3} \delta_\alpha^\nu [q_\lambda, q_\mu^*] + \frac{4}{3} \delta_\nu^\alpha [q_\lambda^*, q_\mu^*] + \frac{4}{3} \delta_\lambda^\nu [q_\mu, q_\lambda^*] +$$

$$+ \frac{4}{3} \delta_\alpha^\nu [q_\lambda^*, q_\mu^*] + \frac{4}{3} \delta_\nu^\alpha [q_\lambda^*, q_\mu^*] + \frac{4}{3} \delta_\lambda^\nu [q_\mu, q_\lambda^*].$$ \hspace{1cm} (7)

Here we have suppressed the momentum variable $k$ and spin variable $\alpha$. The quantities $q_\lambda$ and $r^\lambda$ are expressed as

$$q_\lambda = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} t^\lambda_\alpha, \hspace{1cm} r^\lambda = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} s^\lambda_\alpha$$ \hspace{1cm} (1')

with the reducible representation given by Green. The commutation relations satisfied by $t^\lambda_\alpha$ and $s^\lambda_\alpha$ are
\[
[t_{\alpha\alpha}, t^{*\beta\beta}]_+ = [s^{\alpha\alpha}, s^{*\beta\beta}]_+ = \delta^\beta_\mu,
\]
\[
[t_{\alpha\alpha}, t_{\beta\beta}]_+ = [s^{\alpha\alpha}, s^{\beta\beta}]_+ = [t_{\alpha\alpha}, s^{*\beta\beta}]_+ = [t_{\alpha\alpha}, s^{\beta\beta}]_+ = 0
\] (8)

for identical suffix \(\alpha\) and

\[
[t_{\alpha\alpha}, t^{*\beta\beta}]_- = [s^{\alpha\beta}, s^{*\beta\beta}]_- = 0,
\]
\[
[t_{\alpha\alpha}, t_{\beta\beta}]_- = [s^{\alpha\beta}, s^{\beta\beta}]_- = [t_{\alpha\alpha}, s^{*\beta\beta}]_- = [t_{\alpha\alpha}, s^{\beta\beta}]_- = 0
\] (8')

for \(\alpha \neq \beta\). The origin of the unorthodox statistics is in the fact that \(t_{\lambda\alpha}\) and \(s^{\lambda\alpha}\) commute with each other for \(\alpha \neq \beta\). It is possible, however, to make use of ordinary fermion fields for sub-triplets with the assumption that the vacuum is doubly degenerated and is described as

\[
\Phi_0 = \left( \begin{array}{c} \phi^{(+)}_0 \\ \phi^{(-)}_0 \end{array} \right)
\] (9)

If Pauli matrices \(\omega_\alpha\) operating on the vacuum are introduced, the variables of the triplet field are written as

\[
q_\lambda = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} \omega_\alpha f_{\lambda\alpha}, \quad p^\mu = \frac{1}{\sqrt{3}} \sum_{\alpha=1}^{3} \omega_\alpha g^{\lambda\alpha}, \quad \quad \quad \quad \quad (2')
\]

where \(f_{\lambda\alpha}\) and \(g^{\lambda\alpha}\) satisfy the commutation relations,

\[
[f_{\mu\alpha}, f^{*\nu\beta}]_+ = [g^{\nu\beta}, g^{*\mu\alpha}]_+ = \delta^{*\nu\beta}_\mu\delta_{\alpha\nu}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (10)
\]

and all other anticommutators vanish. We call those fermion fields the sub-triplet fields. The fundamental triplet field is expressed as a mixture of the sub-triplet fields.

The introduction of Pauli matrices enables us to select the states obeying the ordinary statistics from various states composed of the triplet. Such states are restricted to the ones which do not contain Pauli matrix in their expression. The examples of those states up to the third order of field variables are shown below:

\[
[q^{*\mu}, q^{*\nu}]_+ \Phi_0 = \frac{2}{3} \sum_{\alpha=1}^{3} f^{*\mu\alpha} f^{*\nu\alpha} \Phi_0,
\]
\[
[r^{*\mu}, r^{*\nu}]_+ \Phi_0 = \frac{2}{3} \sum_{\alpha=1}^{3} g^{*\mu\alpha} g^{*\nu\alpha} \Phi_0,
\]
\[
[q^{*\mu}, r^{*\nu}]_+ \Phi_0 = \frac{2}{3} \sum_{\alpha=1}^{3} f^{*\mu\alpha} g^{*\nu\alpha} \Phi_0
\] (11)

and
\[
\begin{align*}
[[q^{*\lambda}, q^{*\mu}], q^{*\nu}] \Phi_0 &= \frac{4i}{3\sqrt{3}} \sum_{\alpha, \beta, \gamma=1}^3 \epsilon_{\alpha\beta\gamma} f^{*\lambda\alpha} f^{*\mu\beta} f^{*\nu\gamma} \Phi_0, \\
[[r^*_\lambda, r^*_\mu], r^*_\nu] \Phi_0 &= \frac{4i}{3\sqrt{3}} \sum_{\alpha, \beta, \gamma=1}^3 \epsilon_{\alpha\beta\gamma} q^{*\lambda\alpha} g^{*\mu\beta} g^{*\nu\gamma} \Phi_0, \\
[[q^{*\lambda}, q^{*\mu}], r^*_\nu] \Phi_0 &= \frac{4i}{3\sqrt{3}} \sum_{\alpha, \beta, \gamma=1}^3 \epsilon_{\alpha\beta\gamma} f^{*\lambda\alpha} f^{*\mu\beta} g^{*\nu\gamma} \Phi_0, \\
[[q^{*\lambda}, r^*_\mu], r^*_\nu] \Phi_0 &= \frac{4i}{3\sqrt{3}} \sum_{\alpha, \beta, \gamma=1}^3 \epsilon_{\alpha\beta\gamma} g^{*\lambda\alpha} g^{*\mu\beta} g^{*\nu\gamma} \Phi_0.
\end{align*}
\]

(12)

The two vacuums are completely separated each other in those states obeying the ordinary statistics. Thus the introduction of para-statistics for the triplet does not necessarily bring in a contradiction with the observed phenomena, if the Hamiltonian of the system is invariant under the transformation,

\[
U_\alpha = \exp \left[ -\frac{i}{2} \theta_\alpha \omega_\alpha \right], \quad \alpha = 1, 2, 3.
\]

(13)

§3. Charge mixture and presentation of model

It is shown in the last section that the triplet field \((q_\alpha, r^\lambda)\) obeying para-Fermi statistics is expressed by the sum of three sets of sub-triplet fields \((f_{\lambda\alpha}, g^{\alpha\sigma})\) obeying Fermi statistics. This suggests that a state composed of the triplet is a mixture of the states formed by sub-triplets, and such a state is gemisch in the sense of von Neumann.\(^{30}\)

As briefly discussed in §1, if we define electric and hyper charges for sub-triplets \(f_{\lambda\alpha}\) giving the values \((Q_\alpha+1, Q_\alpha, Q_\alpha)\) and \((Y_\alpha, Y_\alpha, Y_{\alpha-1})\) respectively, under the condition the sum of \(Q_\alpha\) is \(-1\) and that of \(Y_\alpha\) is \(+1\), the triplet is a mixture of the eigenstates of both charges unless \(Q_\alpha\) and \(Y_\alpha\) are independent of \(\alpha\). When the sub-triplets have integral charges, there are three physical quantities for which the triplet is an eigenstate. They are the third component of isospin \(I_3\), strangeness \(S\) and triplet number \(Z\). The triplet state is not an eigenstate of electric charge \(Q\), hypercharge \(Y\) and baryon number \(N\), and merely the expectation values are given for these quantities. They are

\[
\begin{align*}
\langle Q \rangle &= \frac{2}{3}, \quad \langle Q \rangle_2 = \langle Q \rangle_3 = -\frac{1}{3}, \\
\langle Y \rangle &= \frac{1}{3}, \quad \langle Y \rangle_2 = -\frac{2}{3}.
\end{align*}
\]
\[
\langle N \rangle_z = \langle q^{*1} \phi_0 | N | q^{*1} \phi_0 \rangle = \langle N \rangle = \langle N \rangle = \frac{1}{3}.
\]  

(14)

Thus Nakano-Nishijima-Gell-Mann formula holds for the triplet state as a relation among the expectation values,

\[
\begin{align*}
\langle Q \rangle &= I_3' + 1/2 \cdot \langle Y \rangle, \\
\langle Y \rangle &= S' + \langle N \rangle.
\end{align*}
\]

(15)

Here \( I_3' \) and \( S' \) are eigenvalues of the third component of isospin and strangeness respectively. Using the relation (15) and the eigenvalue of triplet number \( Z' \), we can attribute the origin of the mixture to one quantity which is denoted by \( R \):

\[
\begin{align*}
\langle Q \rangle &= I_3' + 1/2 \cdot (S' + Z') - \langle R \rangle, \\
\langle Y \rangle &= S' + Z' - 2\langle R \rangle, \\
\langle N \rangle &= Z' - 2\langle R \rangle.
\end{align*}
\]

(16)

The values of those quantities are tabulated in Tables I and II.

---

**Table I. The values for the triplet states.**

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>( I_3 )</th>
<th>( S )</th>
<th>( Z )</th>
<th>( \langle Q \rangle )</th>
<th>( \langle Y \rangle )</th>
<th>( \langle N \rangle )</th>
<th>( \langle R \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q^{*1} \phi_0 )</td>
<td>1/2</td>
<td>0</td>
<td>1</td>
<td>2/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>( q^{*2} \phi_0 )</td>
<td>-1/2</td>
<td>0</td>
<td>1</td>
<td>-1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>( q^{*3} \phi_0 )</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1/3</td>
<td>-2/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

---

**Table II. The values for the sub-triplet states.**

<table>
<thead>
<tr>
<th>( f^{*11} \phi_0 )</th>
<th>( f^{*12} \phi_0 )</th>
<th>( f^{*13} \phi_0 )</th>
<th>( f^{*21} \phi_0 )</th>
<th>( f^{*22} \phi_0 )</th>
<th>( f^{*23} \phi_0 )</th>
<th>( f^{*31} \phi_0 )</th>
<th>( f^{*32} \phi_0 )</th>
<th>( f^{*33} \phi_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_3 )</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>( S )</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( Z )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \langle Q \rangle )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \langle Y \rangle )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \langle N \rangle )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \langle R \rangle )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Different circumstances from the one-body triplet states discussed
above appear for many body states. There exist eigenstates of charge among many body states. In actual experiment, the measurement of electric charge is performed at some stage intentionally or unintentionally, thus the arrangement is made to prepare eigenstates of electric charge. Stating in another way, only the eigenstates are realized. Therefore, it is quite natural in our model that the one body triplet state is not observed. The eigenstates of electric charge are confined to the following three classes of states for two and three body cases:

\[
[q^{*\mu}, r^{*\nu}] - \Phi_0 = \frac{2}{3} \sum_{s=1}^3 f^{*\mu\alpha} g^{*\alpha}_\nu \Phi_{0s} ,
\]

\[
[[q^{*\lambda}, q^{*\mu}],[q^{*\nu}]]. \Phi_0 = \frac{4i}{3 \sqrt{3}} \sum_{s, \beta, \gamma = 1}^3 \epsilon_{\alpha \beta \gamma} f^{*\lambda \alpha} f^{*\mu \beta} f^{*\nu \gamma} \Phi_{0s} ,
\]

\[
[[r^{\mu}, r^{\nu}],[r^{\nu}]]. \Phi_0 = \frac{4i}{3 \sqrt{3}} \sum_{s, \beta, \gamma = 1}^3 \epsilon_{\alpha \beta \gamma} g^{*\alpha}_\mu g^{*\alpha}_\nu g^{*\beta}_\nu \Phi_{0s} ,
\]

All of them obey the ordinary statistics as can be seen from Eqs. (11) and (12). The first class which obeys Bose statistics takes the values -1, 0 or 1 for electric charge. The eigenvalues of electric charge for the second (third) class, which obeys Fermi statistics, are -1, 0, 1 or 2 (-2, -1, 0 or 1). The baryon number is 0 for the first class and 1 (-1) for the second (third) class. The first class corresponds to mesons, and a state of the second (third) class is observed as a baryon (antibaryon) or its excited state.

§4. Group theoretical consideration on the model

The introduction of para-statistics for the triplet field is needed to investigate the symmetry concerning spin and that of unitary spin simultaneously. We shall discuss the model with \(U(6)\) symmetry in this section, since the case of \(U(3) \times SU(2)\) can be deduced easily from the case of the higher symmetry \(U(6)\).

We change the meaning of the suffix \(\lambda\) of \(q_\lambda\) and \(r^\lambda\) to include the spin variable besides the unitary spin variable, \(q_\lambda \rightarrow q_\lambda\) and \(r^\lambda \rightarrow r^\lambda\) then the range of \(\lambda\) becomes from 1 to 6 instead of from 1 to 3. The generator of the group \(U(6)\) is given by

\[
G^\nu_\mu = \frac{3}{2} \left\{ [q^{*\nu}, q_\mu] - [r^{*\nu}, r^{*\mu}] \right\},
\]

or

\[
= \sum_{s=1}^3 (f^{*\nu\alpha} f^{*\mu\alpha} - g^{*\nu\alpha} g^{*\mu\alpha}) .
\]
The transformation properties of field variables are determined by the following commutators of them with the generator,

$$ [G^x, q^x] = -\delta^x_q q^x, \quad [G^x, r^x] = \delta^x_{r^x}, $$

or

$$ [G^x, f_{x\alpha}] = -\delta^x_{f_{x\alpha}}, \quad [G^x, g^{x\alpha}] = \delta^x_{g^{x\alpha}}. \quad (21) $$

The Hamiltonian of the system should be invariant under the rotation of $\omega$-space (13) and the transformation of $U(6)$ at least in the first approximation. We assume the following form for the Hamiltonian:

$$ H = H_0 + H_1, $$

$$ H_0 = \frac{1}{2} \sum_s \sum_{\mu=1}^6 E_s \{[q^*\mu(k), q_\mu(k)] - [r^*\mu(k), r^\mu(k)]\}, $$

$$ H_1 = \frac{1}{4} \sum_{i, i', \mu, \nu=1}^6 \delta^i(i_1 + k_2 - k_3 - k_4) G(k_1, k_2, k_3, k_4) $$

$$ \times [q^*\mu(k_4) + r^\mu(-k_4), q_\nu(k_4) + r^\nu(-k_4)] - [q^*\nu(k_4) + r^\nu(-k_4), $$

$$ q_\mu(k_4) + r^\mu(-k_4)] \ldots. \quad (22) $$

The relativistic effect and the mass difference among the triplet violate $U(6)$ symmetry. Although the Hamiltonian is invariant under $U(6)$, not all of the eigenstates of it can be realized. The requirement that the realizable states should be the eigenstates of electric charge eliminates the states which form the bases of the irreducible representations of $U(6)$ with dimensions 6, 6, 15, 20, 21, 70 etc. The surviving states belong to the representations with dimensions 1, 35, 56 and their direct products.

Since the triplet field expressed by three sets of sub-triplet fields, we can define a group $U(3)$ which operates on the suffix denoting each set. Its generator is written as

$$ L^\mu_{\alpha} = \sum_{i=1}^6 (f^{*\mu\alpha} f_{i\alpha} - g^{i\alpha} g^{*\mu}). \quad (23) $$

The Hamiltonian (22) is invariant under the group $U(3)$, because it satisfies

$$ [L^\mu_{\alpha}, H] = 0. \quad (24) $$

We can form a group $U(18)$, which contains $U(6) \times U(3)$ as its subgroup, with its generator,

$$ G_{\mu\alpha}^{\beta} = f^{*\nu\alpha} f_{\mu\nu} - g^{*\mu\alpha} g^{\nu\alpha}. \quad (25) $$

But it is shown that our Hamiltonian (22) is not invariant under the
group $U(18)$.

The requirement for the realizable states mentioned above can be formulated with the language of $U(3)$ introduced here. The realizable states should be singlet states of the group $U(3)$ over the triple sets of sub-triplets. This means that there is no accidental degeneracy originated from the $U(3)$ symmetry for the realizable irreducible representation of $U(6)$, even when the state is interpreted as the composite system of sub-triplets. Hence the realizable states up to three body systems are

$$1 + 35: \quad [q^{*\alpha}, r^{*}_\nu] \cdot \Phi_0, \quad (17')$$

$$56: \quad [[q^{*\lambda}, q^{*\mu}], q^{*\nu}] \cdot \Phi_0, \quad (18')$$

and

$$\bar{56}: \quad [[r^{*}_\lambda, r^{*}_\mu], r^{*}_\nu] \cdot \Phi_0. \quad (19')$$

Because of the appearance of $R$, which is equal to $L^\perp_3$ in Eq. (23), in the definition of electric charge, the electromagnetic interaction cannot be invariant under all of the transformations of (23), and some variables other than the triplet field quantities are needed to write down the interaction Hamiltonian. Then, the interaction with magnetic field $A_t$ takes the form in non-relativistic approximation,

$$H_{mag} = \sum_{i=1}^3 \sum_{l=1}^2 \frac{1}{2} \sum_{\lambda, \alpha} \left\{ \right\}

\times \left\{ \right\}

- \frac{1}{4} \left\{ \right\}

\left\{ \right\}.

(26)

Here we have again separated the suffix $\lambda$ to two parts, $\lambda$ for unitary spin and $a$ for spin. The charge operator is given by

$$Q = \frac{3e}{2} \sum_{k} \sum_{a=1}^3 \left\{ \right\}

- \frac{1}{4} \left\{ \right\}

+ \frac{1}{4} \left\{ \right\}

= e \sum_{k} \sum_{a=1}^3 \left\{ \right\}.

(27)
\[ \sum_{k=1}^{s} \left( f^{a \lambda \sigma}(k) f_{\lambda \sigma}(k) - g^{a \lambda \sigma}(k) g_{\lambda \sigma}(k) \right) \]. \hspace{1cm} (27)

Then the magnetization is expressed as

\[ M_i = \frac{3}{2} \mu_0 \sum_{k, a, b=1}^{s} \left\{ [q^{a \lambda \sigma}(k), q_{ib}(k)] - [r^{a \lambda \sigma}(k), r_{ib}(k)] \right\} \]

\[ - \frac{1}{4} \sum_{k=1}^{s} \left[ [\omega_3, q^{a \lambda \sigma}(k)]_+, [\omega_3, q_{ib}(k)]_+ \right] \]

\[ + \frac{1}{4} \sum_{k=1}^{s} \left[ [\omega_3, r^{a \lambda \sigma}(k)]_+, [\omega_3, r_{ib}(k)]_+ \right] \} \langle \sigma_i \rangle^b_a . \] \hspace{1cm} (28)

The expectation value of the operator in baryon state gives the magnetic moment of the baryon. Relations among the values for various baryons coincide with the results obtained in the usual theory with \( U(6) \) symmetry. For instance, the following relation can be easily checked:

\[ \langle p \uparrow | M_3 | p \uparrow \rangle = -\frac{3}{2} \langle n \uparrow | M_3 | n \uparrow \rangle . \] \hspace{1cm} (29)

\section{5. Discussions}

Various attempts have been made to avoid fractional electric charge or baryon number appearing in quark model. Here we have proposed a model in which the appearance of the fractional charge is attributed to the mixed nature of the triplet state. Our triplet corresponds to quarks, but it is not an eigenstate of electric and hyper charge. Only the expectation value of the charges can be calculated for it. The nonexistence of a particle with fractional charge in observed state is naturally understood by para-Fermi statistics which governs the triplet. Those peculiar properties in observable quantities and statistics suggest that the triplet has a structure belonging to so far unknown sub-level. The sub-triplets in our model appear as the reflection of such a structure.

Our fundamental triplet has close analogy to the fundamental triplet in Sakata model in the sense that both of them have three linearly independent quantum numbers, the third component of isospin, strangeness and triplet number. The difference is in the fact that our triplet number cannot be identified with baryon number. Actually our triplet has no definite baryon number, integral or fractional.

We can go further along the analogy. In the Nagoya model, the lepton-baryon symmetry is expressed symbolically by the use of \( B^+ \) matter,
\[ \langle \nu B^+ \rangle = p, \quad \langle eB^+ \rangle = n, \quad \langle \mu B^+ \rangle = \Lambda, \]

where \( \nu \) is some combination of \( \beta \)-neutrino (Paulino) \( \nu_e \) and \( \mu \)-neutrino (Sakatino) \( \nu_\mu \). Baryon number +1 and lepton number −1 can be assigned to \( B^+ \) matter. We employ three kinds of \( B \) matter for a similar correspondence in our model:

\[
\psi_1 = \frac{1}{\sqrt{3}} \left[ \omega_1 \langle \nu_1 B^+ \rangle + \omega_2 \langle \nu_1 B'^+ \rangle + \omega_3 \langle \nu_1 \overline{B}^+ \rangle \right],
\]

\[
\psi_2 = \frac{1}{\sqrt{3}} \left[ \omega_1 \langle e B^+ \rangle + \omega_2 \langle e B'^+ \rangle + \omega_3 \langle e \overline{B}^+ \rangle \right],
\]

\[
\psi_3 = \frac{1}{\sqrt{3}} \left[ \omega_1 \langle \mu B^+ \rangle + \omega_2 \langle \mu B'^+ \rangle + \omega_3 \langle \mu \overline{B}^+ \rangle \right].
\]

These \( B \) matters, \( B^+ \), \( B'^+ \) and \( \overline{B}^+ \), have baryon number \( (1, 1, -1) \) and lepton number \( (-1, -1, -1) \). Though \( B^+ \) and \( B'^+ \) have same quantum numbers, they should be considered as independent entities. Baryons and their decuplet excited states are expressed as \( \langle l_1 l_2 l_3 B^+ B'^+ \overline{B}^+ \rangle \), where \( l_i \) denotes leptons. Since leptons obey Fermi statistics, \( B \) matter part should be responsible for the violation of Pauli principle, otherwise the above expression does not correspond to 1/2+ octet and 3/2+ decuplet states. Such an effect cannot be expected to arise just by the addition of \( (B^+ B'^+ \overline{B}^+) \) to three leptons. That is the reason why we have started from the triplet obeying para-statistics.

Our model in which the triplet is treated as a mixture of the sub-triplets suggests that the sub-triplets may play more essential role in the structure of elementary particles. But the arguments made in this paper has shown that the triplet can be considered as a fundamental constituent as long as the investigations are restricted to strong interaction. This is because the dual vacuum has isotropic property under the transformation \( U_\alpha \). For the electromagnetic interaction where the invariance under the transformation \( U^2 \), \( L^2_\alpha \), is broken, the triplet cannot be considered as the most fundamental material.

The expression (2) for the triplet field quantity has been utilized throughout this paper. We shall discuss the meaning of the expression (1) briefly. It can be easily seen that the Hamiltonians (22) and (26) conserve the number of particles belonging to each set denoted by \( \alpha \). Thus there is no difficulty to interpret the fields \( t_{\alpha \alpha} (s^{\alpha \alpha}) \) as the fermion triplets belonging to families specified by \( \alpha \). The commutation relations (8) and (8') are compatible with the interpretation. Though the Hamiltonian (22) is invariant under \( U(3) \) over the suffix \( \alpha \) in this case too, the commutation relations do not allow such an invariance. These
are the main difference from the other case discussed in this paper. In the case of expression (2), the system is invariant under $U(3)$, but it is necessary to introduce the twofold structure of the vacuum besides the sub-triplets obeying the ordinary Fermi statistics. The dual nature of the vacuum allows us to write down the electromagnetic interaction without using the field variables for sub-triplets explicitly.

The present situation of the theory of elementary particles seems to compel us to pay a profound consideration on the existence of some kind of sub-level hidden under the observed physical world. The appearance of fractional charge may arises from formal application of complicated mathematics without due consideration on the deep structure of Nature.

The model proposed here may offer a clue to the sub-level on which we have not yet any concrete knowledge, though the investigation of dynamical problems will be indispensable for the clarification of the true nature of the sub-level.

Acknowledgement

The authors express their sincere thanks to Prof. H. Yukawa for his kind encouragement.

References

3) H. Goldberg and Y. Ne’eman, Nuovo Cim. 27 (1963) 1.
6) T. Okayama, Prog. Theor. Phys. 7 (1952), 517.
   H. S. Green, Phys. Rev. 90 (1953), 270.
   S. Kamefuchi and J. Strathdee, Nucl. Phys. 42 (1963), 166.
   This new kind of statistics in quantum field theory was first investigated by T. Okayama in 1952. The name “para-statistics” is taken solely for its simplicity, though it may be more suitable to call it Okayama-statistics.
   M. Gell-Mann, Phys. Rev. 92 (1953), 833.
A Varitian triplet model and integral charge

   675.
14) Two distinguishable neutrinos have been predicted by W. Pauli (Proc. of Solvay
   Congress (1933), 324) for the beta-decay and by S. Sakata and T. Inoue (Prog.
   Theor. Phys. 1 (1946), 143) for the pi-mu decay. We propose the name of "Paulino" and
   "Sakatino" for the beta-neutrino and the mu-neutrino respectively in honour of their
   pioneer works. The name of "Neutrino" is used to call both particles together.