A Wightman-Function Approach to Relativistic Complex-Ghost Field Theory

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(Received September 19, 2006)

Relativistic complex-ghost field theory is covariantly formulated in terms of Wightman functions. The Fourier transform of the 2-point Wightman function of a complex-ghost pair is explicitly calculated, and the spontaneous violation of its Lorentz invariance is compared with that of the corresponding Feynman integral.

§1. Introduction

Abe and the present author have proposed and established a general method for finding the solution to quantum field theory, formulated as a canonical operator formalism, in the Heisenberg picture.1) The totality of full-dimensional (anti-) commutators for the field operators is obtained as a solution to the q-number Cauchy problem constructed from field equations and equal-time (anti-)commutation relations. Next, the representation of this operator solution is found by constructing the set of Wightman functions. In contrast to the situation in axiomatic quantum field theory, however, the Wightman functions thus obtained do not, in general, satisfy the norm-positivity condition of the state-vector space. That is, the natural framework of Lagrangian quantum field theory is an indefinite-metric theory.

One of the striking properties of an indefinite-metric theory is that, within it, the eigenvalues of a hermitian operator are not necessarily real; that is, we generally encounter complex-energy eigenvalues of the Hamiltonian. This fact contradicts the usual spectral condition postulated in axiomatic quantum field theory. The “energy-positivity condition” has also been required to hold in the above-mentioned method for finding the solution to quantum field theory in terms of Wightman functions. Specifically, we have required every Wightman function $W(x_1, \ldots, x_n)$ to be a boundary value of an analytic function from the lower half-planes of the time differences $x_1^0 - x_2^0, \ldots, x_{n-1}^0 - x_n^0$; this property follows from the positivity of the energy. Hence, if there are states with complex energy, we must examine whether or not the energy-positivity condition is still applicable to our method.

States with complex energy should not appear in the physical world; that is, such states must be unphysical. For this reason, they are called “complex ghosts”.2) A single complex ghost has zero norm, and it cannot appear in the final state if the initial state has real energy, because of energy conservation. However, if a complex ghost exists, there is always a complex-conjugate ghost, and a state consisting of a complex ghost and its conjugate has real energy but can have negative norm. Hence,
the energy conservation condition cannot forbid the appearance of a complex-ghost pair, and for this reason, it was conjectured that the unitarity of the physical S-matrix would always be violated. Indeed, it has been confirmed that this conjecture holds in the Lee model.

In 1969-1970, however, Lee and Wick\textsuperscript{[3,4]} (and Lee\textsuperscript{5}) pointed out that the unitarity of the physical S-matrix is not violated in the relativistic complex-ghost field theory. This follows from the fact that because the relativistic energy takes the form of the square root of the sum of the mass squared and the spatial momentum squared, the complex conjugation of a complex mass does not necessarily imply the complex conjugation of the corresponding energy; therefore, the unitarity cut due to the complex-ghost pair is absent. This wonderful result, however, is valid only in the case that Lorentz invariance is violated, as proved by the present author\textsuperscript{6} and explicitly confirmed by Gleeson, Moore, Rechenberg and Sudarshan\textsuperscript{7} (GMRS) subsequently. Lee and Wick\textsuperscript{8} attributed the violation of Lorentz invariance to the fact that the spatial momentum is real while the energy is complex, and suggested to make possible the use of complex spatial momenta by adopting the S-matrix-theoretical rule previously proposed by Cutkosky, Landshoff, Olive and Polkinghorn.\textsuperscript{9} The violation of Lorentz invariance, however, is not a direct consequence of the reality of the spatial momenta, because a complex energy is encountered only on the mass shell, and because quantum field theory is formulated on the basis of off-mass-shell quantities. Indeed, the present author\textsuperscript{10} showed that relativistic complex-ghost field theory can be formulated manifestly covariantly without using complex masses in the action, and that it is actually Lorentz invariant at the operator level.

The purpose of the present paper is to reformulate the relativistic complex-ghost field theory in terms of Wightman functions. Its manifestly covariant formulation presented previously is based on momentum-space considerations;\textsuperscript{10} After introducing creation and annihilation operators for real-mass fields explicitly, the complex-ghost fields were constructed by means of the Bogoliubov transformation. In the present paper, we directly solve the Cauchy problem for the 4-dimensional commutators and then construct the Wightman functions without employing the momentum space. We show that the energy-positivity condition remains effective in spite of the presence of complex ghosts. The 2-point Wightman functions of the fundamental fields are shown to be Lorentz invariant. The violation of Lorentz invariance, however, is found for the 2-point Wightman functions of composite fields. We explicitly calculate the Fourier transform of the 2-point Wightman function of a complex-ghost pair, and compare it with the corresponding Feynman integral calculated by GMRS. Although no spectral representation exists in the present case, the expression for the Wightman function is found to be similar to the spectral function for the Feynman amplitude.

The present paper is organized as follows. In §2, we apply the Wightman-function approach to the complex-ghost theory and construct the Wightman functions explicitly. In §3, we review the spontaneous violation of Lorentz invariance in the Feynman integral involving a complex-ghost pair intermediate state. In §4, we explicitly calculate the Fourier transforms of the 2-point Wightman functions of a complex-ghost pair and of two complex ghosts. The final section is devoted to
discussion.

§2. Formulation of the theory

We start with the Lagrangian density\textsuperscript{5,10)}

\[ \mathcal{L} = \frac{1}{2} \sum_{j=1}^{2} (-1)^{j-1} (\partial^{\mu} \phi_j \cdot \partial_{\mu} \phi_j - m^2 \phi_j^2) - \gamma m \phi_1 \phi_2, \]  

(2.1)

where \( \phi_1 \) and \( \phi_2 \) are hermitian scalar fields and \( m \) and \( \gamma \) are positive constants.

The field equations are

\[ (\Box + m^2) \phi_1 + \gamma m \phi_2 = 0, \]
\[ (\Box + m^2) \phi_2 - \gamma m \phi_1 = 0, \]  

(2.2)

and the non-vanishing equal-time commutation relations are

\[ [\partial_0 \phi_j(x), \phi_j(y)]_{x^0=y^0} = -(-1)^{j-1} i \delta(x - y). \]  

(2.3)

We write

\[ [\phi_j(x), \phi_k(y)] \equiv i \Delta_{jk}(x - y) \]  

(2.4)

and

\[ \Delta \equiv \text{matrix}(\Delta_{jk}). \]  

(2.5)

We then have the Cauchy problem

\[ (\Box^x + m^2 + i \gamma m \sigma_2) \Delta(x - y) = 0, \]  

(2.6)

together with

\[ \Delta(x - y)|_{x^0=y^0} = 0, \]
\[ \partial_0^x \Delta(x - y)|_{x^0=y^0} = -\sigma_3 \delta(x - y), \]  

(2.7)

where \( \sigma_i \) denotes the Pauli matrix.

We diagonalize (2.6) by using the unitary matrix \( U \equiv (i + \sigma_1)/\sqrt{2} \). Then, because \( U \sigma_2 U^{-1} = \sigma_3 \) and \( U \sigma_3 U^{-1} = -\sigma_2 \), we obtain

\[ (\Box^x + m^2 + i \gamma m \sigma_3) \hat{\Delta}(x - y) = 0, \]  

(2.8)

together with

\[ \hat{\Delta}(x - y)|_{x^0=y^0} = 0, \]
\[ \partial_0^x \hat{\Delta}(x - y)|_{x^0=y^0} = \sigma_2 \delta(x - y), \]  

(2.9)

where \( \hat{\Delta} \equiv U \Delta U^{-1} \).

We define the complex-mass \( \Delta \) function through the Cauchy problem

\[ (\Box^x + M^2) \Delta(x - y; M^2) = 0, \]  

(2.10)
\[ \Delta(x - y; M^2)|_{x^0 = y^0} = 0, \]
\[ \partial_0 \Delta(x - y; M^2)|_{x^0 = y^0} = -\delta(x - y), \] (2.11)

where \( M^2 \) is a complex number such that \( \Re M^2 > 0 \). The explicit expression for the complex-mass \( \Delta \) function is

\[ \Delta(\xi; M^2) = \frac{1}{(2\pi)^3} \int dp \frac{\sin(p\xi - E_p\xi^0)}{E_p} \]
\[ = -\frac{1}{2\pi} e(\xi^0) \left[ \delta(\xi^2) - \frac{M^2}{2} \theta(\xi^2) \frac{J_1(M\sqrt{\xi^2})}{M\sqrt{\xi^2}} \right], \] (2.12)

where \( E_p \equiv \sqrt{M^2 + p^2} \) and \( J_1 \) denotes a Bessel function. The complex-mass \( \Delta \) function is, of course, Lorentz invariant, though its absolute value increases exponentially as \( \xi^2 \to \infty \).

The solution to the Cauchy problem (2.8) with (2.9) is given by

\[ \hat{\Delta}(x - y) = i\sigma_+ \Delta(x - y; M^2) - i\sigma_- \Delta(x - y; M^{*2}), \] (2.13)

where \( \sigma_\pm \equiv (\sigma_1 \pm i\sigma_2)/2 \) and \( M^2 \equiv m^2 + i\gamma m \). Hence we have

\[ \Delta(x - y) = U^{-1} \hat{\Delta}(x - y)U = \sigma_3 \Re \Delta(x - y; M^2) - \sigma_1 \Im \Delta(x - y; M^2). \] (2.14)

We next consider the representation of the above operator solution. First, we introduce the vacuum \(|0\rangle\) and set \langle 0|0 \rangle = 1. The 1-point Wightman functions, \langle 0|\phi_j(x)|0 \rangle, are set equal to zero so as to be consistent with translational invariance and the field equations. Then the 2-point truncated Wightman functions are the same as the untruncated ones.

The 2-point Wightman functions, \langle 0|\phi_j(x)\phi_k(y)|0 \rangle, must be constructed so as to be consistent with the commutator functions given by (2.14) and with the energy-positivity condition. We define the complex-mass “positive-energy” \( \Delta \) function by

\[ \Delta^{(+)}(\xi; M^2) = \frac{1}{(2\pi)^3} \int dp \frac{\exp i(p\xi - E_p\xi^0)}{2E_p}. \] (2.15)

Note that the momentum integral is convergent, because \( \Im E_p \) tends to zero as \( |p| \to \infty \). Evidently, \( \Delta^{(+)}(\xi; M^2) \) is Lorentz invariant and has the properties

\[ \Delta^{(+)}(\xi; M^2) + \Delta^{(+)}(-\xi; M^2) = i\Delta(\xi; M^2), \] (2.16)
\[ \Delta^{(+)}(-\xi; M^2) = [\Delta^{(+)}(\xi; M^{*2})]^*. \] (2.17)

Furthermore, because \( \Re E_p > 0 \), \( \Delta^{(+)}(\xi; M^2) \) is a boundary value of an analytic function from the lower half-plane of \( \xi^0 \), in conformity with the requirement of the energy-positivity condition.

From (2.14), we find that the 2-point (truncated) Wightman functions are given by

\[ \langle 0|\phi_j(x)\phi_j(y)|0 \rangle = (-1)^{j-1-1} \frac{1}{2} \{ \Delta^{(+)}(x - y; M^2) + \Delta^{(+)}(x - y; M^{*2}) \}. \] (2.18)
\[ \langle 0 | \phi_1(x) \phi_2(y) | 0 \rangle = \langle 0 | \phi_2(x) \phi_1(y) | 0 \rangle = -\frac{1}{2i} [\Delta^+(x - y; M^2) - \Delta^+(x - y; M^{*2})]. \]  

All \( n \)-point truncated Wightman functions with \( n > 2 \) vanish, because all multiple commutators vanish. Thus, all Wightman functions of the fundamental fields are Lorentz invariant.

§3. Violation of Lorentz invariance

As we have shown in §2, the theory is strictly Lorentz invariant as far as the Wightman functions of the fundamental fields are concerned. Quite interestingly, however, Lorentz invariance is not valid for composite fields. In this section, we briefly review the results obtained previously for the 1-loop Feynman integral involving a complex-ghost pair intermediate state.

In the Fourier representation of the Feynman propagator \( \Delta_F(\xi; M^2) \) [but not \( \Delta_F(\xi; M^{*2}) \)], we need to introduce a complex contour \( C \) that runs above the pole located at \( p_0 = \sqrt{M^2 + \mathbf{p}^2} \), in spite of the fact that it lies in the upper half-plane. Therefore, in order to write down the momentum-space Feynman integral, we necessarily encounter position-space integrals of the type

\[ f(k_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx^0 e^{i(k_0 - \lambda)x^0}, \]  

where \( \lambda \) is a complex number. Naively, this integral is divergent and mathematically meaningless. By adopting a Gaussian adiabatic hypothesis in defining the Dyson S-matrix, however, we can show that (3.1) should be defined in the sense of the “complex \( \delta \)-function”,\(^{10} \) \( f(k_0) = \delta(k_0 - \lambda). \)\(^{\ast} \) The complex \( \delta \)-function is defined by extending the definition of the Schwinger distribution in the following way. Let \( \varphi(k_0) \) be a test function that is an arbitrary function holomorphic in an appropriate strip domain about the real axis. Then we define \( \delta(k_0 - \lambda) \) through the relation

\[ \int_{-\infty}^{\infty} dk_0 \varphi(k_0) \delta(k_0 - \lambda) = \frac{1}{2\pi i} \oint dk_0 \frac{\varphi(k_0)}{k_0 - \lambda}, \]  

where the contour goes around \( k_0 = \lambda \) in the anticlockwise direction. Of course, if \( \lambda \) is real, the complex \( \delta \)-function reduces to the ordinary \( \delta \)-function.

Now, the Feynman integral involving a complex-ghost pair intermediate state is given by

\[ F_{MM^*}(p) = \frac{1}{(2\pi)^4 i} \int dq \int_C dq_0 \frac{1}{(q^2 - M^2)((p - q)^2 - M^{*2})}, \]  

where we understand that its ultraviolet divergence is appropriately subtracted. The integrand of (3.3) has four poles in the \( q_0 \) plane. The contour \( C \) runs from \(-\infty\) to \(+\infty\), passing below the two poles \(-\mathcal{E}_q\) and \( p_0 - \mathcal{E}^{*}_{-q} \) and above the other two poles,

\(^{\ast} \) This concept was introduced by the present author\(^{11} \) in 1958.
$E_q$ and $p_0 + E_{p-q}^*$. Then, carrying out the contour integration of (3.3), we obtain

$$F_{MM^*}(p) = \frac{1}{2(2\pi)^3} \int dq \left( \frac{1}{E_q} + \frac{1}{E_{p-q}^*} \right) \frac{1}{p_0^2 - (E_q + E_{p-q}^*)^2}. \quad (3.4)$$

When $q$ runs over the whole three-dimensional space, the locus of

$$p_0 = E_q + E_{p-q}^*, \quad (3.5)$$

which corresponds to a unitarity cut in the real-mass case, sweeps out a 2-dimensional fish-shaped domain $D$ in the $p_0$ plane. Its boundary curve $\Gamma$ intersects the real axis at only one point, $p_0 = 2\Re E_{p/2}$. Evidently, the quantity $s_{\text{min}}$, defined as

$$s_{\text{min}} \equiv 4(\Re E_{p/2})^2 - p^2, \quad (3.6)$$

is not Lorentz invariant [$s_{\text{min}} < (M + M^*)^2$ for $p \neq 0$].

GMRS explicitly carried out the momentum integration given in (3.4). Their result is*)

$$F_{MM^*}(p) = \frac{1}{16\pi^2} \Re \int_{\Gamma_s} ds' \left[ \frac{\sqrt{p^2(M^2 - M^*)^2}}{s' \sqrt{s' + p^2}} + \sqrt{(s' - (M + M^*)^2)(s' - (M - M^*)^2)} \right], \quad (3.7)$$

where $s \equiv p_0^2 - p^2$, and the contour $\Gamma_s$ is the lower half of the image of $\Gamma$ under the mapping $s' = p_0^2 - p^2$; it runs from $s_{\text{min}}$ to $\infty$.

§4. Wightman function of a complex-ghost pair

In this section, we explicitly calculate the Fourier transform of the 2-point Wightman function of a complex-ghost pair. Its formal expression is given by

$$W_{MM^*}(p) \equiv \int d^4 \xi \Delta^{(+)}(\xi; M^2) \Delta^{(+)}(\xi; M^*) e^{ip\xi}. \quad (4.1)$$

Of course, naively (4.1) is meaningless, because the absolute value of $\Delta^{(+)}(\xi; M^2)$ is an exponentially increasing function of $\xi^0$. Substituting (2.15) into (4.1) and carrying out one of the spatial momentum integrations, we obtain

$$W_{MM^*}(p) = \frac{1}{(2\pi)^3} \int dq \frac{1}{4E_qE_{p-q}^*} \int_{-\infty}^{\infty} d\xi^0 \exp[i(p_0 - E_q - E_{p-q}^*)\xi^0]. \quad (4.2)$$

Now, the energy factor inside the exponential in (4.2) is complex for $p \neq 0$, and thus understood naively, the integration over $\xi^0$ is meaningless, as stated above. We

*) In (3.7), an error (an overall factor of 1/2) has been corrected. This error appears in (A3) of Ref. 7.)
define the integral over $\xi^0$ by means of the complex $\delta$-function, as in §3. That is, in accordance with (3.2), we consider

$$
\int_{-\infty}^{\infty} dp_0 \varphi(p_0)W_{M,M^*}(p) = \oint dp_0 \varphi(p_0)I(p),
$$

(4.3)

where

$$
I(p) \equiv \frac{-i}{4(2\pi)^3} \int dq \frac{1}{E_qE_{p-q}(p_0 - E_q - E_{p-q}^*)}.
$$

(4.4)

Without loss of generality, we can use the coordinate system defined by $p = (0, 0, p_3 > 0)$. Then, employing cylindrical coordinates for $q$ and setting $\rho^2 = q_1^2 + q_2^2$, we obtain

$$
I(p) = \frac{-i}{16\pi^2} \int_{-\infty}^{\infty} dq_3 \int_{0}^{\infty} \frac{dq_0}{\rho^2 E(\rho^2, q_3)E^*(\rho^2, -q_3)[p_0 - E(\rho^2, q_3) - E^*(\rho^2, -q_3)]},
$$

where

$$
E(\rho^2, q_3) \equiv \sqrt{M^2 + \rho^2 + \left(\frac{p_3}{2} + q_3\right)^2}.
$$

(4.5)

By transforming the integration variable $\rho^2$ into the complex variable

$$
q_0 = E(\rho^2, q_3) + E^*(\rho^2, -q_3),
$$

(4.7)

the integral reduces to the very simple one

$$
I(p) = \frac{-i}{16\pi^2} \int_{-\infty}^{\infty} dq_3 \int_{\alpha(q_3)}^{\infty} \frac{dq_0}{q_0(p_0 - q_0)},
$$

(4.8)

where

$$
\alpha(q_3) \equiv E(0, q_3) + E^*(0, -q_3).
$$

(4.9)

The integration over $q_0$ is easily carried out, and we obtain

$$
I(p) = \frac{-i}{16\pi^2 p_0} \int_{-\infty}^{\infty} dq_3 \log \frac{\alpha(q_3) - p_0}{\alpha(q_3)}.
$$

(4.10)

Transforming the integration variable $q_3$ into $\alpha = \alpha(q_3)$ for $q_3 \geq 0$ and $\alpha = \alpha(-q_3) = [\alpha(q_3)]^*$ for $q_3 \leq 0$, we have

$$
I(p) = \frac{-i}{16\pi^2 p_0} \left( \int_{\Gamma(+) + \Gamma(+)^*} d\alpha \frac{dq_3(\alpha)}{d\alpha} \log(\alpha - p_0) - \log \alpha \right)
= \frac{i}{16\pi^2 p_0} \left[ \int_{\Gamma(+) + \Gamma(+)^*} d\alpha q_3(\alpha) \left( \frac{1}{\alpha - p_0} - \frac{1}{\alpha} \right) + c. \right.
$$

(4.11)

Here, $q_3(\alpha)$ is the inverse function of $\alpha = \alpha(q_3)$; explicitly, it is given by

$$
q_3(\alpha) = \frac{p_3(M^2 - M^*2) + \alpha \sqrt{\alpha^2 - p_3^2 - (M + M^*)^2}(\alpha^2 - p_3^2 - (M - M^*)^2)}{2(\alpha^2 - p_3^2)},
$$

(4.12)
where the sign of the square root has been chosen in such a way that \( q_3 = \alpha/2 \) when \( M = 0 \) and \( q_3 > p_3/2 \), as seen from (4.9).\(^1\) The quantity \( c \) in (4.11) is the integration constant multiplied by the coefficient, that is, it is given by \( c = -i/16\pi^2 \). Furthermore, the contour \( \Gamma^{(\pm)} \) is the image of the positive real axis under the mapping \( \alpha = \alpha(q_3) \) for \( q_3 \geq 0 \); that is, it is the upper boundary curve of \( D \) introduced in \( \S 3 \). Both \( \Gamma^{(\pm)} \) and \( \Gamma^{(\pm)*} \) can be deformed into the same real interval, and this yields

\[
I(p) = \frac{i}{8\pi^2 p_0} \int_{2\Re p/2}^{\infty} d\alpha q_3(\alpha) \left( \frac{1}{\alpha - p_0} - \frac{1}{\alpha} \right) + c. \tag{4.13}
\]

Now, substituting (4.13) into (4.3), we obtain

\[
\int_{-\infty}^{\infty} dp_0 \varphi(p_0) W_{MM^*}(p) = \int dp_0 \varphi(p_0) \left[ \frac{i}{8\pi^2} \int_{2\Re p/2}^{\infty} d\alpha \frac{q_3(\alpha)}{\alpha(\alpha - p_0)} + c \right] \tag{4.14}
\]

\[
= \int_{2\Re p/2}^{\infty} d\alpha \varphi(\alpha) \frac{1}{4\pi} \frac{q_3(\alpha)}{\alpha}.
\]

Then, rewriting the right-hand side of (4.14) as

\[
\int_{-\infty}^{\infty} dp_0 \varphi(p_0) \frac{1}{4\pi} \frac{q_3(p_0)}{p_0} \theta(p_0 - 2\Re p/2), \tag{4.15}
\]

we find

\[
W_{MM^*}(p) = \frac{1}{4\pi} \frac{q_3(p_0)}{p_0} \theta(p_0 - 2\Re p/2). \tag{4.16}
\]

From (4.16) with (4.12), therefore, our final result is

\[
W_{MM^*}(p) = \frac{1}{8\pi} \left[ \sqrt{p^2(M^2 - M^{*2})} + \sqrt{(s - (M + M^*)^2)(s - (M - M^*)^2)} \right] \theta(s - s_{\text{min}}) \tag{4.17}
\]

for the general value of \( p \).

The above result reveals that \( W_{MM^*}(p) \) is not only Lorentz non-invariant but also complex valued. In contrast to the real-mass case, in which the 2-point Wightman function is the discontinuity function of the Feynman amplitude, according to the Cutkosky rule, \( W_{MM^*}(p) \) has no direct connection to \( F_{MM^*}(p) \). Nevertheless, comparing (4.17) with (3.7), we find that the relationship between the two expressions is similar to that in the real-mass case.

For completeness, here we calculate the Wightman function of two complex

\(^*\) For \( \Re \alpha^2 < (M + M^*)^2 + p_3^2 \), the square root in (4.12) should take a value lying in the lower half-plane, so as to realize \( q_3(\alpha) = 0 \) for \( \alpha = 2\Re p/2 \).
ghosts,

\[ W_{MM}(p) \equiv \int d^4 \xi \left[ \Delta^{(+)}(\xi; M^2) \right]^2 e^{ip\xi} \]

\[ = \frac{1}{(2\pi)^3} \int dq \frac{1}{4E_qE_{p-q}} \int_{-\infty}^{\infty} d\xi^0 \exp[i(p_0 - E_q - E_{p-q})\xi^0]. \]  \hspace{1cm} (4.18)

The calculation is carried out in almost the same way as the above calculation, except for the fact that the locus of \( p_0 = E_q + E_{p-q} \) does not intersect the real axis in the present case. Therefore, the final result cannot be expressed in terms of an ordinary function or distribution. Then, introducing a "complex \( \theta \)-function", defined by

\[ \theta_c(k_0 - \lambda) \equiv \int_{-\infty}^{\infty} dk_0 \delta_c(k_0' - \lambda), \]  \hspace{1cm} (4.19)

we can write the final result in the following way:

\[ W_{MM}(p) = \frac{1}{8\pi} \sqrt{\frac{s - 4M^2}{s}} \theta_c(s - 4M^2). \]  \hspace{1cm} (4.20)

Of course, as \( \Im M \to 0 \), both (4.17) and (4.20) tend to the well-known expression for the Wightman function (which equals the discontinuity function of the Feynman amplitude along the real axis) in the equal real-mass case.

\section{Discussion}

In the present paper, we have successfully applied a method for finding the solution to quantum field theory in the Heisenberg picture to the covariant formulation of complex-ghost theory. In spite of the fact that there is a complex energy spectrum, we find that the energy-positivity condition can be applied without a problem.

We have studied the spontaneous violation of Lorentz invariance encountered in relativistic complex-ghost theory from the viewpoint of the Wightman function. We have explicitly calculated the Fourier transforms of the 2-point Wightman functions of a complex-ghost pair and of two complex ghosts. The former indeed does exhibit the violation of Lorentz invariance. In spite of the fact that the spectral representation for the Feynman amplitude does not exist, the relation between the Feynman amplitude and the corresponding Wightman function is quite similar to that in the real-mass case.

We have seen that the introduction of the complex \( \delta \)-function is essential in deriving the results stated above. From (2.18), we have

\[ \langle 0 | [\phi_j(x)]^2 [\phi_j(y)]^2 | 0 \rangle = \frac{1}{2} [\Delta^{(+)}(x - y; M^2) + \Delta^{(+)}(x - y; M^*^2)]^2. \]  \hspace{1cm} (5.1)

According to (4.17) and (4.20), therefore, the Fourier transform of (5.1) is neither Lorentz invariant nor expressible in terms of ordinary distributions. This conclusion is rather surprising in view of the fact that the Lagrangian density (2.1) is very simple and explicitly contains no complex numbers. This suggests that it is very difficult to avoid the appearance of the extraordinary situation encountered above \textit{a priori} in the general framework of Lagrangian quantum field theory.
References

1) For a review, see N. Nakanishi, Prog. Theor. Phys. 111 (2004), 301. Further references are contained therein.


