Group Theoretical Analysis of the Vortex Lattice States in the Attractive Hubbard Model

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This paper describes a group theoretical classification of superconducting states (vortex lattice states) in a uniform magnetic field of the extended Hubbard model with on-site attraction ($U < 0$), or nearest-neighbor attraction ($V < 0$) on a two-dimensional square lattice. Using symmetries of the magnetic translation and the tetragonal rotation of the system, we obtain invariance groups of vortex lattice states. When the magnetic flux $\phi$ through a unit cell of the crystal lattice is $\phi_0/p^2$ ($\phi_0 = ch/2e$ is the flux quantum, $p$ is an integer) we obtained three types of symmorphic tetragonal vortex lattice states, in which the center of point rotations is at a lattice site of the crystal lattice, corresponding to the irreducible representation of the tetragonal symmetry $D_4$. For these three states, locations and winding numbers of vortex lines are determined by the symmetries of their invariance group. From these three symmorphic states we derived three (two) types of non-symmorphic vortex lattice states in which the center of point rotations is at the center of a plaquette (link) of the crystal lattice. We show that $d$-wave, $s$-wave and $p$-wave local order parameters inevitably coexist due to the invariance magnetic translation group $L$ of the vortex lattice state. From the irreducible projective representations (two-dimensional) of $L$, we show that all energy bands of the quasiparticles are at least doubly degenerate.

§1. Introduction

In the absence of a magnetic field, bulk superconductivity (SC) does not break translational symmetry of crystal lattice as a whole, even if it is anisotropic SC. In this paper we consider the following problems from a group theoretical point of view:

1. Why does a superconducting state break the translational symmetry of a crystal lattice in the presence of a uniform magnetic field?
2. What kinds of translational symmetry can remain in superconducting states (vortex lattice states) in the presence of a uniform magnetic field?
3. What kinds of rotational symmetries can there be in superconducting states with a definite remaining translational symmetry?
4. Does the concept of a local symmetric order parameter (OP), such as a $d$-wave, $s$-wave, etc., work in vortex lattice states?
5. What types of symmetry properties are there for the quasiparticles of the vortex lattice states?

Several articles ¹⁻⁴ have been devoted to the symmetry analysis on the vortex lattice state in a magnetic field. However, these studies employ a Ginzburg-Landau

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theory and do not take into account the crystal lattice symmetry. Hence a microscopic theory for the symmetry analysis of the vortex lattice state is necessary and important.

For this purpose we take the extended Hubbard model with an attractive interaction on a two-dimensional square lattice for illustrative purposes. In a previous paper, $^5,^6$ we gave a group theoretical classification of superconducting states in the extended Hubbard model with an attractive interaction on a two-dimensional square lattice using the Hartree-Fock-Bogoliubov equation in the absence of a magnetic field. In the previous paper, $^5,^6$ we derived two singlet SCs ($s$-wave and $d$-wave) and eight triplet SCs including three non-unitary states.

Recently much attention has been focused on a single vortex and vortex lattice states of the extended Hubbard model in the presence of magnetic fields. $^8$-$^11$ Thus it is important to classify the SC state of the extended Hubbard model in the presence of a magnetic field.

The group of translational symmetry of the system in a magnetic field is the group of the magnetic translation (we denote by $\mathbf{T}$) instead of usual translation. Thus we cannot straightforwardly apply the previous group theoretical analysis $^5$ to this case. Noting that the translation of crystal lattice acts as a projective representation on the space of the mean-field Hamiltonian $H_m$, we have obtained a group theoretical classification of SC states (vortex lattice states) in the presence of a magnetic field for the extended Hubbard model. We classify the vortex lattice states by the invariance group of the mean-field Hamiltonian $H_m$. The invariance group is expressed by in the form $\mathbf{PL}$, where $\mathbf{P}$ is a subgroup of $\mathbf{D}_4\Phi$ and $\mathbf{L}$ (referred to as the invariance magnetic translation group of the vortex state) is a subgroup $\mathbf{T}\Phi$ ($\Phi$ is the global gauge transformation).

Detailed study has been done in the case that the magnetic flux $\phi$ through a unit cell of the crystal lattice is $\phi_0/p^2$ ($\phi_0 =\hbar/2e$ is the flux quantum, $p$ is an integer). In this case we obtained three types of symmorphic tetragonal vortex lattice states with invariance group $G_t = \mathbf{P}L(\mathbf{i} = 1, 3, 5)$, in which the center of point rotations is at a lattice point of the crystal lattice. We determined the locations and winding numbers of vortex lines for these states using symmetries of $G_t$. In these cases, using the irreducible projective representation (IPR) of $\mathbf{L}$, we obtain the standard form of the mean-field Hamiltonian $H_m$ in terms of irreducible bases of $\mathbf{L}$. It is shown that all bands of quasi-particles are necessarily at least doubly degenerate as all IPRs of $\mathbf{L}$ are of two dimensions.

From the $\mathbf{L}$ symmetry of the vortex state, it is shown that local symmetric OPs such as $s$-, $d$- and $p$-wave OPs always coexist. Then concepts of local $d$-wave OP or $s$-wave OP are not good for the characterization of the vortex state.

From these three symmorphic states we derived three (two) types of non-symmmorphic vortex lattice states in which the center of point rotations is at the center of a plaquette (link) of the crystal lattice.

This paper is organized as follows: In §2 we give the model Hamiltonian and its symmetry properties. In §3 we give general symmetry properties of the mean-field Hamiltonian and order parameter. In §4 we discuss possible translational symmetries of the superconducting states. In §5 we discuss possible rotational symmetries of the
superconducting states. In §6 we give the classification of the tetragonal vortex lattice in the case of $\phi = \phi_0/p^2$. In §7 we show that an order parameter (OP) with a definite symmetry, such as an s-wave, d-wave and p-wave, on a site $(m, n)$ must coexist from the $L$ invariance of the vortex state. In §8 we give bases of an irreducible projective representation (IPR) of the invariance magnetic translation group $L$. In §9 we give the standard form of the mean field Hamiltonian in terms of irreducible bases of IPR of $L$. It is shown that all bands of quasiparticle are doubly degenerate. In §10 we give a summary and discussion.

The notation used in this paper follows that of our previous paper, \textsuperscript{5) which is Schönflies notation for point groups and their elements.\textsuperscript{12)}

\section*{§2. Model Hamiltonian and its symmetry}

We consider the two-dimensional Hubbard Hamiltonian

\begin{align}
H &= H_0 + H_1, \\
H_0 &= \sum_{(ij)s} (-t_{ij} - \mu \delta_{ij}) a_{i_s}^\dagger a_{js}, \\
H_1 &= U \sum_i n_{i\uparrow} n_{i\downarrow} + V \sum_{(i,j) s,s'} \sum n_{i,s} n_{j,s'},
\end{align}

where $(ij)$ denotes summation over nearest-neighbor pairs in the square lattice, and $U$ and $V$ denote on-site and nearest-neighbor interactions. We assume that the type II limit of superconductors, where the magnetic field can be regarded as constant. The magnetic field $(B|e_z)$ enters the Hamiltonian through a vector potential in the Peierls phase of the hopping integral,

\begin{equation}
t_{ij} \Rightarrow t \exp \left\{ -\frac{e}{\hbar c} \int_i^j A(r) dr \right\},
\end{equation}

where $-e < 0$ is the charge of electron. We take the symmetric gauge

\begin{equation}
A(r) = \frac{1}{2} (B \times r) = \left( -\frac{1}{2} By, \frac{1}{2} Bx, 0 \right),
\end{equation}

where $B = (0, 0, B)$ is a magnetic field. Labelling a site by $(m, n)$, we have

\begin{align}
\int_{(m,n)}^{(m+1,n)} A(r) dr &= -\frac{B}{2} n a a = -\frac{1}{2} B a^2 n, \\
\int_{(m,n)}^{(m,n+1)} A(r) dr &= \frac{1}{2} B m a a = \frac{1}{2} B a^2 m,
\end{align}

where $a$ is the lattice constant of the square lattice. Defining $\phi = B a^2$ and $\phi_0 = \frac{\hbar}{2e}$, we obtain the hopping integral as follows:

\begin{align}
t_{(m,n),(m+1,n)} &= t \exp \left\{ i \frac{\pi \phi}{2 \phi_0} n \right\}, \\
t_{(m,n),(m,n+1)} &= t \exp \left\{ -i \frac{\pi \phi}{2 \phi_0} m \right\}.
\end{align}
Defining \( K = \frac{\pi}{2} \frac{\phi}{\phi_0} \), we have

\[
\begin{align*}
t_{(m,n), (m+1,n)} &= t e^{iKn}, \\
t_{(m,n), (m,n+1)} &= t e^{-iKm}.
\end{align*}
\] (2.6)

From this we obtain \( H \) as follows:

\[
H = -t \sum_{(m,n)s} \left\{ (e^{iKn} a_{(m,n)s}^\dagger a_{(m+1,n)s} + e^{-iKm} a_{(m,n)s}^\dagger a_{(m,n+1)s}) + \text{h.c.} \right\} - \mu \sum_{(m,n)s} a_{(m,n)s}^\dagger a_{(m,n)s} + U \sum_{(m,n)} a_{(m,n)}^\dagger a_{(m,n)}^\dagger a_{(m,n)} a_{(m,n)} + V \sum_{(m,n)ss'} \left\{ a_{(m,n)s}^\dagger a_{(m,n)s} a_{(m+1,n)s'} a_{(m+1,n)s'} + a_{(m,n)s}^\dagger a_{(m,n+1)s'} a_{(m+1,n)s'} \right\}.
\] (2.7)

The symmetry group of the system is given by

\[
G_0 = (e + tC_{2s})C_4TST\Phi,
\] (2.8)

where \( T \) is the group of the magnetic translation, \(^{14}\) consisting of the element \( T(Mae_x + Naey)(M,N = \text{integer}) \) such that

\[
\begin{align*}
T(Mae_x + Naey) \cdot a_{(m,n)s}^\dagger &= e^{iK(Mn-Nm)} a_{(m,M,n+N)s'}^\dagger, \\
T(Mae_x + Naey) \cdot a_{(m,n)s} &= e^{-iK(Mn-Nm)} a_{(m,M+n+N)s}.
\end{align*}
\] (2.9)

\( C_4 = (e, C_{4s}^+, C_{2s}, C_{4s}^-) \) is the four-fold rotation group. Its generator's action on the Fermion operator is given by

\[
C_{4s}^+ \cdot a_{(m,n)s}^\dagger = a_{(-n,m)s'}^\dagger.
\] (2.10)

The action of \( tC_{2s} \) (\( t \) is the time reversal) \(^5\) on the Fermion operator is

\[
\begin{align*}
tC_{2s} \cdot (fa_{(m,n)}^\dagger) &= -f^* a_{(m,-n)}^\dagger, \\
tC_{2s} \cdot (fa_{(m,n)}^\dagger) &= f^* a_{(m,-n)}^\dagger.
\end{align*}
\] (2.11)

where \( f \) is a complex number. \( S \) is the group of the spin rotation (\( SU(2) \)), and \( \Phi \) is the group of global gauge transformation. The action of \( u(e, \theta) \in S \) and \( \phi \in \Phi \) are defined \(^5\), \(^7\) by

\[
\begin{align*}
u(e, \theta) \cdot a_{(m,n)s} &= \sum_{s'} \{ u(e, \theta) \}_{s's} a_{(m,n)s'}, \\
u(e, \theta) \cdot a_{(m,n)s} &= \sum_{s'} \{ u(e, \theta) \}_{s's} a_{(m,n)s'}.
\end{align*}
\] (2.12)
\begin{align}
\hat{\phi} \cdot a_{(m,n)s}^\dagger &= e^{i\phi/2}a_{(m,n)s}^\dagger, \\
\hat{\phi} \cdot a_{(m,n)s} &= e^{-i\phi/2}a_{(m,n)s},
\end{align}

where \( u(e, \theta) \) is a spin rotation by \( \theta \) radian around the \( e \) axis and is given by

\begin{equation}
u(e, \theta) = \cos(\theta/2)1 - i(\sigma \cdot e) \sin(\theta/2),\end{equation}

with the Pauli matrix \( \sigma = (\sigma^x, \sigma^y, \sigma^z) \) and the \( 2 \times 2 \) unit matrix \( 1 \). \( \hat{\phi} \) is a phase change by \( \phi \).

Although continuous translation \( T((m+x)e_x + (n+y)e_y)(0 < x, y < 1) \) does not have a physical meaning in this model, for later purposes we introduce here the fictitious translation \( T(xae_x + yae_y) \) and the Fermion operators \( a_{(m+x,n+y)s}^\dagger, a_{(m+x,n+y)s} \) such that

\begin{align}
T(xae_x + yae_y) \cdot a_{(m,n)s}^\dagger &= e^{iK(xn-ym)}a_{(m+x,n+y)s}^\dagger, \\
T(xae_x + yae_y) \cdot a_{(m,n)s} &= e^{-iK(xn-ym)}a_{(m+x,n+y)s}, \\
T(xae_x + yae_y)T(x'e_x + y'e_y)a_{(m,n)s}^\dagger &= e^{iK(x'y'-yx')}T((x+x')ae_x + (y+y')ae_y)a_{(m,n)s}^\dagger,
\end{align}

where \( x \) and \( y \) are real numbers and \( m \) and \( n \) are integers. The actions of \( C_{4z}^+ \) and \( tC_{2x} \) on \( fa_{(x,y)s}^\dagger (f \) is a complex number) are defined by

\begin{align}
C_{4z}^+ \cdot fa_{(x,y)s}^\dagger &= fa_{(-y,x)s}^\dagger, \\
tC_{2x} \cdot fa_{(x,y)s}^\dagger &= -f^*a_{(x,-y)s}^\dagger, \\
tC_{2x} \cdot fa_{(x,y)s} &= f^*a_{(x,-y)s}^\dagger.
\end{align}

From (2.15) and (2.16) we obtain

\begin{align}
T(xae_x + yae_y)C_{4z}^+ &= C_{4z}^+T(yae_x - xae_y), \\
T(xae_x + yae_y)C_{2z} &= C_{2z}T(-xae_x - yae_y), \\
T(xae_x + yae_y)tC_{2x} &= tC_{2x}T(xae_x - yae_y).
\end{align}

The \( 2\pi/4 \) and \( 2\pi/2 \) rotations around \( (x, y) \) (we denote them by \( C_{4z}^+(x, y) \) and \( C_{2z}(x, y) \)) are defined by

\begin{align}
C_{4z}^+(x, y) &= T(xae_x + yae_y)C_{4z}^+T(-xae_x - yae_y) \\
&= C_{4z}^+T(yae_x - xae_y)T(-xae_x - yae_y) \\
&= e^{-iK(x^2+y^2)}C_{4z}^+T((y-x)ae_x - (x+y)ae_y),
\end{align}

and

\begin{align}
C_{2z}(x, y) &= T(xae_x + yae_y)C_{2z}T(-xae_x - yae_y) \\
&= C_{2z}T(-xae_x - yae_y)T(-xae_x - yae_y) \\
&= C_{2z}T(-2xae_x - 2yae_y).
\end{align}
In the same way we define
\[ tC_{2x}(x, y) = T(xae_x + yae_y)tC_{2x}T(-xae_x - yae_y) \]
\[ = tC_{2x}T(xae_x - yae_y)T(-xae_x - yae_y) \]
\[ = e^{i2Kxy}tC_{2x}T(-2ye_y). \] (2.20)

The phase factors \( e^{-iK(x^2+y^2)} \) and \( e^{i2Kxy} \) in (2.18) and (2.20) should be understood to appear when group elements act on the space \( W(a^\dagger) \) of Fermion creation operators. Hereafter, phase factors in relations among group elements carry the above meaning.

§3. Mean-field Hamiltonian and its invariance group

In this section we give a short review of the symmetry properties of the Hartree-Fock-Bogoliubov equation.

The general mean-field Hamiltonian \( H_m \) is given by
\[ H_m = H_{m1} + H_{m2}, \]
\[ H_{m1} = \sum_{i,j} x_{i,j} a_i^\dagger a_j, \]
\[ H_{m2} = \sum_{i,j} (y_{i,j} a_i^\dagger a_j^\dagger + y_{ji}^* a_i a_j), \] (3.1)

where \( i \) and \( j \) represent single particle states including spin states, and \( x_{i,j} \) and \( y_{ij} \) are complex numbers which satisfy Hermite and antisymmetric conditions: \( x_{i,j} = x_{j,i}^*, y_{i,j} = -y_{ji}. \) In general a superconducting state is characterized \(^7\) by a subgroup \( G_s \) of the original symmetry group \( G_0 \) of the system which breaks the global gauge symmetry. \( H_m \) corresponding to the state is invariant under the action \( G_s: \)
\[ g \cdot H_m = H_m \quad \text{for} \ g \in G_s. \] (3.2)

We call \( G_s \) the invariance group of \( H_m. \) Inversely we can define invariance group \( G_s \) of \( H_m \) by
\[ G_s = \{ g \in G_0 | g \cdot H_m = H_m \}. \] (3.3)

When \( G_s \) is the invariance group of \( H_m \) and \( g \in G_0, \) since
\[ (gG_sg^{-1}) \cdot (g \cdot H_m) = g \cdot G_s \cdot H_m = (g \cdot H_m), \] (3.4)
\( (gG_sg^{-1}) \) is the invariance group of \( (g \cdot H_m). \) Thus a conjugate subgroup \( (gG_sg^{-1}) \) is physically equivalent to \( G_s \) as \( (g \cdot H_m) \) is physically equivalent to \( H_m. \) According to a previous paper, \(^7\) the generalized density matrix is invariant with respect to \( G_s: \)
\[ \langle (g \cdot a_i^\dagger)(g \cdot a_j) \rangle^{(s)} = \langle a_i^\dagger a_j \rangle, \]
\[ \langle (g \cdot a_i^\dagger)(g \cdot a_j) \rangle^{(s)} = \langle a_i^\dagger a_j^\dagger \rangle, \]
\[ \langle (g \cdot a_i)(g \cdot a_j) \rangle^{(s)} = \langle a_i a_j \rangle. \] (3.5)
for \( g \in G_s \), where \( \langle A \rangle \) denotes the expectation value of \( A \) with respect to the mean-field Hamiltonian \( H_m \),

\[
\langle A \rangle = \frac{\text{Tr}(e^{-\beta H_m} A)}{\text{Tr}(e^{-\beta H_m})},
\]

and \( (\ast) \) denotes the complex conjugate in the case of an anti-unitary \( g \), such as \( tC_{2x} \). Thus we can classify superconducting states by listing invariance groups \( G_s \in G_0 \) (up to the equivalence class of conjugation) such that \( G_s = \{ g \in G_0 \mid g \cdot H_m = H_m \} \) for \( H_m \neq 0 \). Note that not all subgroups \( G_i \) of \( G_0 \) can be invariance groups of \( H_m \). For example, when \( y_{i,j} \neq 0 \) for some \( i \) and \( j \), \( H_m \) cannot have an invariance group \( G_s \) involving the entire \( T \).

\section*{§4. Magnetic translation symmetry}

In general, the mean-field Hamiltonian \( H_m \) of a singlet (SC) is expressed by

\[
H_m = H_{mN} + H_{mS},
\]

\[
H_{mN} = \sum_s \sum_{(m,n)} \sum_{(m',n')} x(m,n : m',n') a_{(m,n)s}^\dagger a_{(m',n')s},
\]

\[
H_{mS} = \sum_{(m,n)} \sum_{(m',n')} \{ y(m,n : m',n')(a_{(m,n)\uparrow}^\dagger a_{(m',n')\downarrow}^\dagger - a_{(m,n)\downarrow}^\dagger a_{(m',n')\uparrow}^\dagger) + \text{h.c.} \}.
\]

\[(4-1)\]

In this section we consider what type of subgroup of \( T\Phi \) can be included in the invariance group of the mean-field Hamiltonian \( H_m \). First we consider a subgroup of \( T \). A subgroup of \( T \) is a magnetic superlattice translation group and consists of elements \( T(M(p_1 a e_x + p_2 a e_y) + N(p_1' a e_x + p_2' a e_y)) \), where \( M, N, p_1, p_2, p_1', \) and \( p_2' \) are integers. The basis vectors of the superlattice are \( l_1 = p_1 a e_x + p_2 a e_y \) and \( l_2 = p_1' a e_x + p_2' a e_y \). We denote this magnetic translation subgroup by \( T(l_1, l_2) \). From (2.9), we obtain

\[
T(\text{N}l_2)T(Ml_1)a_{(m,n)s}^\dagger a_{(m',n')s} = T(Ml_1 + Nl_2)a_{(m,n)s}^\dagger a_{(m',n')s},
\]

\[
T(Ml_1)T(\text{N}l_2)a_{(m,n)s}^\dagger a_{(m',n')s} = T(Ml_1 + Nl_2)a_{(m,n)s}^\dagger a_{(m',n')s},
\]

\[
T(\text{N}l_2)T(Ml_1)a_{(m,n)\uparrow}^\dagger a_{(m',n')\downarrow}^\dagger = e^{2iKM_N(p_1'p_2 - p_2'p_1)}T(Ml_1 + Nl_2)a_{(m,n)\uparrow}^\dagger a_{(m',n')\downarrow}^\dagger,
\]

\[
T(Ml_1)T(\text{N}l_2)a_{(m,n)\uparrow}^\dagger a_{(m',n')\downarrow}^\dagger = e^{2iKM_N(p_1'p_2 - p_2'p_1)}T(Ml_1 + Nl_2)a_{(m,n)\uparrow}^\dagger a_{(m',n')\downarrow}^\dagger.
\]

\[(4-2)\]

Note that \( |(p_1p_2' - p_2p_1')|a^2 \) is the area of the unit cell of the superlattice generated by \( l_1 \) and \( l_2 \). In order for the magnetic superlattice group \( T(l_1, l_2) \) to be included in the invariance group of \( H_m \), the following relations must hold:

\[
T(Ml_1)H_m = H_m,
\]

\[
T(Nl_2)H_m = H_m,
\]
Then we obtain
\[ e^{2iK(p_1p'_2 - p_2p'_1)}H_m = H_m. \] (4.4)
Thus we have
\[ 2K(p_1p'_2 - p_2p'_1) = 2\pi J, \] (4.5)
where \( J \) is an integer. We then obtain
\[ \frac{\phi}{\phi_0} = \frac{2J}{p_1p'_2 - p_2p'_1}. \] (4.6)
In this case, \( T(l_1, l_2) \) can be included in the invariance group of \( H_m \). Thus for small \( \frac{\phi}{\phi_0} \), that is, in the case of a small magnetic field, the area of the unit cell of the superlattice must be large. Then translational symmetry must be broken to a great extent.

When
\[ \frac{\phi}{\phi_0} = \frac{J}{p_1p'_2 - p_2p'_1}, \] (4.7)
where \( J \) is an odd integer instead of (4.6), we can consider a subgroup of \( T\Phi \) which can be included in the invariance group of \( H_m \) as follows. We rearrange the phase of the magnetic translation by\(^{13}\)
\[ L(Ml_1 + Nl_2)a^\dagger_{(m,n)s} \equiv e^{i\frac{\pi}{2}(MN+M+N)}T(Ml_1 + Nl_2)a^\dagger_{(m,n)s}. \] (4.8)
Then we have
\[ L(Ml_1)L(Nl_2)a^\dagger_{(m,n)s}a^\dagger_{(m',n')s'} = L(Ml_1 + Nl_2)a^\dagger_{(m,n)s}a^\dagger_{(m',n')s'}. \] (4.9)
Thus the group \( L(l_1, l_2) \) consisting of \( L(Ml_1 + Nl_2) \) can be included in the invariance group of \( H_m \). We call \( L(l_1, l_2) \) or \( T(l_1, l_2) \) the invariance magnetic translation group.

In the following, we consider the case \( J = 1 \):
\[ \frac{\phi}{\phi_0} = \frac{1}{p_1p'_2 - p_2p'_1}. \] (4.10)
In this case the magnetic flux through the primitive magnetic unit cell (PMUC) with area \( |l_1 \times l_2| \) is \( \phi_0 \). For \( \phi/\phi_0 = 1/q \), where \( q \) is an integer, there can be various kinds of sets of \((p_1, p_2), (p'_1, p'_2)\) such that
\[ q = p_1p'_2 - p_2p'_1. \] (4.11)
For example, in the case \( q = 8 \) we have sets of \( \{(p_1, p_2), (p'_1, p'_2)\} \) as follows: \( \{(2, 2), (-2, 2)\}, \{(4, 0), (0, 2)\}, \{(4, 0), (1, 2)\}, \{(4, 0), (2, 2)\}, \{(4, 0), (3, 2)\}, \{(4, 2), (2, 3)\}, \{(3, 1), (1, 3)\}, \) etc. In the case \( q = 9 \) we have sets of \( \{(p_1, p_2), (p'_1, p'_2)\} \) as follows: \( \{(3, 0), (0, 3)\}, \{(3, 0), (1, 3)\}, \{(3, 0), (2, 3)\}, \{(3, 1), (0, 3)\}, \{(4, 1), (3, 3)\}, \) etc. Various magnetic unit cells for \( q = 8 \) and \( 9 \) are displayed in Figs. 1 and 2.
Thus for $\phi/\phi_0 = 1/q$ ($q$: integer), there can be various types of vortex lattice states with the basis vectors $l_1 = p_1a_e + p_2a_e$ and $l_2 = p'_1a_e + p'_2a_e$ such that $|p_1p'_2 - p_2p'_1| = q$. Then, in order to determine which type of vortex states are realized, we must calculate the total free energies for various vortex lattice states and choose the most stable state.

§5. Space point symmetry

In this section we consider the rotational symmetry of the vortex lattice with the basis vector $l_1 = p_1a_e + p_2a_e$ and $l_2 = p'_1a_e + p'_2a_e$. When $q = p^2$ for integer $p$, we can obtain a vortex lattice with tetragonal symmetry for $(p_1, p_2) = (p, 0), (p'_1, p'_2) = (0, p)$. Also when $q = 2p^2$ for integer $p$, we can obtain a vortex lattice with the tetragonal symmetry for $(p_1, p_2) = (p, p), (p'_1, p'_2) = (-p, p)$. In these cases, vortex-lattice states are classified further by tetragonal rotational symmetry.
Table I. Basis vectors of the vortex lattice and point symmetries. $p$, $p_1$ and $p_2$ are integers.

<table>
<thead>
<tr>
<th>Basis vectors</th>
<th>Point Symmetry</th>
<th>comments</th>
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<tbody>
<tr>
<td>$(p,0),(0,p)$</td>
<td>$D_4 = C_4 + tC_{2x}C_4$</td>
<td></td>
</tr>
<tr>
<td>$(p,p),(-p,p)$</td>
<td>$D_4 = C_4 + tC_{2x}C_4$</td>
<td></td>
</tr>
<tr>
<td>$(p_1,p_2),(-p_2,p_1)$</td>
<td>$C_4$</td>
<td>$p_1 \neq p_2, p_1 \neq 0, p_2 \neq 0$</td>
</tr>
<tr>
<td>$(p_1,0),(0,p_2)$</td>
<td>$D_2 = C_2 + tC_{2x}C_2$</td>
<td>$p_1 \neq p_2$</td>
</tr>
<tr>
<td>$(p_1,p_2),(p_1,-p_2)$</td>
<td>$D_2 = C_2 + tC_{2x}C_2$</td>
<td>$p_1 \neq p_2, p_1 \neq 0, p_2 \neq 0$</td>
</tr>
<tr>
<td>$(p_1,p_2),(-p_2,p_2)$</td>
<td>$D_{2a} = C_2 + tC_{2x}C_2$</td>
<td>$p_1 \neq p_2, p_1 \neq 0, p_2 \neq 0$</td>
</tr>
<tr>
<td>other case</td>
<td>$C_2$</td>
<td></td>
</tr>
</tbody>
</table>

$D_4 = (e + tC_{2x})C_4$. Depending on the set of basis vectors, we have various point symmetries, as shown in Table I. For a generic case, the point symmetry of the lattice is $C_2$. Depending on the point symmetry, there are various types of vortex lattice states corresponding to irreducible representations of the group of the point symmetry.

In the next section we consider in detail a square vortex lattice with $D_4 = (e + tC_{2x})C_4$ point symmetry having basis vectors $(p_1,p_2) = (p,0),(p_1',p_2') = (0,p), q = p^2$ as an illustration.

§6. Classification of the square vortex lattice

Hereafter we restrict our consideration to the case $\phi = \phi_0/p^2$ for an illustrative purpose. Then we consider the square vortex lattice states having the invariance magnetic translation group $L(l_1,l_2)$ with basis vectors $l_1 = pae_x, l_2 = pae_y$. We denote $L(l_1,l_2)$ by $L$. First let us consider a symmorphic subgroup $G'_0 \subset G_0$ given by

$$G'_0 = (e + tC_{2x})C_4 LS\Phi.$$  \hfill(6-1)

Cases of non-symmorphic subgroups \(^{19}\) are considered at the end of this section. From (2-10), (2-11) and (4-9), $G'_0$ has a vector representations on the space of the mean-field Hamiltonian $W_m$, where

$$W_m = \{a^\dagger_{(m,n)s}a_{(m',n')s'}a^\dagger_{(m,n)s'}a_{(m',n')s'}, a_{(m,n)s}a_{(m',n')s'}\}_{h.c.}. \hfill(6-2)$$

Here $\{A,B,\cdots\}_{h.c.}$ denotes a vector space over a complex number field spanned by $A,B,$ etc., such that its elements are Hermite. $G'_0$ is a maximal symmorphic subgroup such that its action on $W_m$ is a vector representation. Then, the symmorphic broken symmetry states can be derived from irreducible representations of $G'_0$ over the real number field ($R$-rep). \(^7\) We consider the singlet superconducting states with maximal translational symmetry. These are derived from the identity representation of $LS$. Then the relevant $R$-rep of $G'_0$ is determined by the $R$-rep of $D_4 = (e + tC_{2x})C_4$. The $R$-reps of $D_4$ are $A_1, A_2, B_1, B_2$ and $E$. From each one-dimensional $R$-rep among $A_1, A_2, B_1, B_2$ we obtain one maximal little group (invariance group) and from the two-dimensional $R$-rep $E$, we obtain two maximal little groups using the method used in Ref. 15). The results are summarized in Table II. From the discussion in §3, the generalized density matrix $\langle a^\dagger_{(m,n)s}a_{(m',n')s}\rangle$ and $\langle a_{(m,n)}a^\dagger_{(m',n')}\rangle$ have symmetries
Table II. $R$-rep and invariance group in $D_4LS$.

<table>
<thead>
<tr>
<th>$R$-rep</th>
<th>Invariance group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$:</td>
<td>$G_1 = (e + tC_{2x})(e + C_{4z}^+)(e + C_{2x})LS$</td>
</tr>
<tr>
<td>$A_2$:</td>
<td>$G_2 = (e + \pi tC_{2x})(e + C_{4z}^+)(e + C_{2x})LS$</td>
</tr>
<tr>
<td>$B_1$:</td>
<td>$G_3 = (e + tC_{2x})(e + \pi C_{4z}^+)(e + C_{2x})LS$</td>
</tr>
<tr>
<td>$B_2$:</td>
<td>$G_4 = (e + \pi tC_{2x})(e + \pi C_{4z}^+)(e + C_{2x})LS$</td>
</tr>
<tr>
<td>$E$:</td>
<td>$G_5 = (e + tC_{2x})(e + (\pi/2)C_{4z}^+)(e + \pi C_{2x})LS$</td>
</tr>
<tr>
<td>$E$:</td>
<td>$G_6 = (e + tC_{2x})(e + (\pi/2)C_{4z}^+)(e + \pi C_{2x})LS$</td>
</tr>
</tbody>
</table>

given by $G_i$. As all states have the symmetry of $LS$, for $L(Mpae_x + Npae_y) \in L$, we have

$$
\langle (L(Mpae_x + Npae_y)a^\dagger_{(m,n)s}) (L(Mpae_x + Npae_y)a_{(m',n')s'}) \rangle \\
= \langle (a^\dagger_{(m,n)s}a_{(m',n')s'}) \rangle,
$$

$$
\langle (L(Mpae_x + Npae_y)a_{(m,n)s}) (L(Mpae_x + Npae_y)a_{(m',n')s'}) \rangle \\
= \langle (a_{(m,n)s}a_{(m',n')s'}) \rangle,
$$

(6.3)

and for $u(e_y, \pi) \in S$

$$
\langle (u(e_y, \pi)a^\dagger_{(m,n)s}) (u(e_y, \pi)a_{(m',n')s'}) \rangle = \langle a^\dagger_{(m,n)s}a_{(m',n')s'} \rangle,
$$

$$
\langle (u(e_y, \pi)a_{(m,n)s}) (u(e_y, \pi)a_{(m',n')s'}) \rangle = \langle a_{(m,n)s}a_{(m',n')s'} \rangle.
$$

(6.4)

From (2-9), (4-8), (6-3) and (6-4) we have

$$
\langle a^\dagger_{(m+M,p+n+Np)s}a_{(m'+Mp,n'+Np)s'} \rangle \\
= e^{-i\frac{\pi}{2p} (M(n-n')-N(m-m'))} \langle a^\dagger_{(m,n)s}a_{(m',n')s'} \rangle,
$$

$$
\langle a_{(m+M,p+n+Np)s}a_{(m'+Mp,n'+Np)s'} \rangle \\
= e^{i\pi(MN+M+N)+i\frac{\pi}{2p} (M(n+n')-N(m+m'))} \\
\times \langle a_{(m,n)s}a_{(m',n')s'} \rangle,
$$

(6-5)

$$
\langle a^\dagger_{(m,n)}a_{(m',n')\dagger} \rangle = \langle a^\dagger_{(m,n)}a_{(m',n')\dagger} \rangle, \\
\langle a_{(m,n)}a_{(m',n')\dagger} \rangle = -\langle a_{(m,n)}a_{(m',n')\dagger} \rangle.
$$

(6-6)

From $u(e_z, \pi)$ invariance we obtain

$$
\langle a^\dagger_{(m,n)}a_{(m',n')\dagger} \rangle = \langle a^\dagger_{(m,n)}a_{(m',n')\dagger} \rangle = 0, \\
\langle a_{(m,n)}a_{(m',n')\dagger} \rangle = \langle a_{(m,n)}a_{(m',n')\dagger} \rangle = 0.
$$

(6-7)

For all $G_i$ ($i = 1, \cdots, 6$), OPs have this translational and spin rotational symmetry. Space point rotational symmetry is different for each $G_i$ ($i = 1, \cdots, 6$). We consider six cases separately.

6.1. $G_1$ state

In this case the symmetry of the vortex lattice is given by the subgroup

$$
G_1 = \overline{P}_1 LS, \\
\overline{P}_1 = (e + tC_{2x})C_4.
$$

(6-8)
The mean-field Hamiltonian is invariant under the action of $G_1$. The generalized density matrix is invariant with respect to $G_1$. For example, from $C_{4z}^+ \in C_4$ invariance, we have

\[
\langle (C_{4z}^+ a_{(m,n)s}^\dagger) (C_{4z}^+ a_{(m',n')s'}) \rangle = \langle a_{(m,n)s}^\dagger a_{(m',n')s'} \rangle;
\]

\[
\langle (C_{4z}^+ a_{(m,n)s}) (C_{4z}^+ a_{(m',n')s'}) \rangle = \langle a_{(m,n)s} a_{(m',n')s'} \rangle. \tag{6.9}
\]

From (2.10), this implies

\[
\langle a_{(-n,m)s}^\dagger a_{(-n',m')s'} \rangle = \langle a_{(m,n)s}^\dagger a_{(m',n')s'} \rangle;
\]

\[
\langle a_{(-n,m)s} a_{(-n',m')s'} \rangle = \langle a_{(m,n)s} a_{(m',n')s'} \rangle. \tag{6.10}
\]

Thus from $C_4$ invariance we obtain

\[
\langle a_{(m,n)s}^\dagger a_{(m',n')s'} \rangle = \langle a_{(-n,m)s}^\dagger a_{(-n',m')s'} \rangle
\]

\[
= \langle a_{(-m,-n)s}^\dagger a_{(-m',-n')s'} \rangle
\]

\[
= \langle a_{(n,-m)s} a_{(n',-m')s'} \rangle. \tag{6.11}
\]

\[
\langle a_{(m,n)s} a_{(m',n')s'} \rangle = \langle a_{(-n,m)s} a_{(-n',m')s'} \rangle
\]

\[
= \langle a_{(-m,-n)s} a_{(-m',-n')s'} \rangle
\]

\[
= \langle a_{(n,-m)s} a_{(n',-m')s'} \rangle. \tag{6.12}
\]

In a similar manner, from $tC_{2x}, tC_{2y}, tC_{2a}$ and $tC_{2b}$ invariance we obtain

\[
\langle a_{(m,n)s}^\dagger a_{(m',n')s'} \rangle = \pm \langle a_{(m,-n)s}^\dagger a_{(m',-n')s'} \rangle^*
\]

\[
= \pm \langle a_{(-m,n)s}^\dagger a_{(-m',n')s'} \rangle^*
\]

\[
= \pm \langle a_{(n,m)s}^\dagger a_{(n',m')s'} \rangle^*
\]

\[
= \pm \langle a_{(-n,-m)s}^\dagger a_{(-n',-m')s'} \rangle^*. \tag{6.13}
\]

\[
\langle a_{(m,n)s} a_{(m',n')s'} \rangle = \pm \langle a_{(m,-n)s} a_{(m',-n')s'} \rangle^*
\]

\[
= \pm \langle a_{(-m,n)s} a_{(-m',n')s'} \rangle^*
\]

\[
= \pm \langle a_{(n,m)s} a_{(n',m')s'} \rangle^*
\]

\[
= \pm \langle a_{(-n,-m)s} a_{(-n',-m')s'} \rangle^*. \tag{6.14}
\]

where $+(-)$ corresponds to the case $s = s'(s \neq s')$, and $s$ denotes the opposite spin of $s$ ($\uparrow = \downarrow, \downarrow = \uparrow$). From the definition of (2.13) and Table II, relations (6.11) and (6.13) hold for all $G_i$. From (6.6) and (6.14) we obtain

\[
\langle a_{(m,0)s}^\dagger a_{(m',0)s'} \rangle^* = \langle a_{(m,0)s}^\dagger a_{(m',0)s'} \rangle;
\]

\[
\langle a_{(0,n)s}^\dagger a_{(0,n')s'} \rangle^* = \langle a_{(0,n)s}^\dagger a_{(0,n')s'} \rangle;
\]

\[
\langle a_{(m,m)s}^\dagger a_{(m',m')s'} \rangle^* = \langle a_{(m,m)s}^\dagger a_{(m',m')s'} \rangle;
\]

\[
\langle a_{(m,-m)s}^\dagger a_{(m',-m')s'} \rangle^* = \langle a_{(m,-m)s}^\dagger a_{(m',-m')s'} \rangle. \tag{6.15}
\]
Fig. 3. Primitive order parameters (POPs) at the origin for the case \( l = 2 \). \( o \) denotes on-site OP: \( \langle a_{(i,j)}^\dagger a_{(i,j)} \rangle \), \( \times \) denotes nearest-neighbor OP: \( \langle a_{(i,j)}^\dagger a_{(i+1,j)} \rangle \) or \( \langle a_{(i,j)}^\dagger a_{(i,j+1)} \rangle \).

Thus the above OPs must be real. For the purpose of illustration, hereafter we consider the case \( p = 2l + 1 \), where \( l > 0 \) is an integer. We call the set of lattice points \( (m,n)(m,n = -l, -l+1, \ldots, 0, \ldots, l) \) a primitive magnetic unit cell (PMUC) at the origin (see Fig. 3). From \( C_{2z}, L(pa e_x) \) and \( L(pa e_y) \) invariance, we have

\[
\begin{align*}
\langle a_{(l,0)}^\dagger a_{(l+1,0)} \rangle &= \langle a_{(-l-1,0)}^\dagger a_{(-l,0)} \rangle = 0, \\
\langle a_{(0,0)}^\dagger a_{(0,l+1)} \rangle &= \langle a_{(0,-l-1)}^\dagger a_{(0,-l)} \rangle = 0.
\end{align*}
\]

From (6-5), (6-12) and (6-14), OPs at all sites are obtained from the OPs in the region illustrated in Fig. 3 in the PMUC at the origin. We call these OPs primitive order parameters (POPs). For \( l = 1 \), that is, \( p = 3 \), using (6-15) we obtain a pattern of OPs, which is displayed in Fig. 4. Note that in PMUC at the origin, all on-site order parameters \( \langle a_{(m,n)}^\dagger a_{(m,n)} \rangle \) \( (m, n = -1, 0, 1) \) are real numbers. This comes from (6-15).

As we can see from Fig. 4, the locations of vortex lines 19) are corners of PMUC, e.g. \( (1 + \frac{1}{2}, 1 + \frac{1}{2}) \), and the centers of sides of squares of PMUC, e.g. \( (1 + \frac{1}{2}, 0) \). Now we consider the vortex nature around a point \( (l + \frac{1}{2}, l + \frac{1}{2}) = (p/2, p/2) \) of a corner of PMUC. From (2-18) the \( \frac{2\pi}{4} \) rotation around \( (l + \frac{1}{2}, l + \frac{1}{2}) \) is given by

\[
\begin{align*}
C_{4z}^+(p/2, p/2) &= e^{-iK(p^2/4+p^2/4)}C_{4z}^+T(-pa e_y) \\
&= e^{-i\frac{\pi}{4}}C_{4z}^+T(-pa e_y) \\
&= e^{+i\frac{\pi}{4}}C_{4z}^+L(-pa e_y).
\end{align*}
\]
Although $C_{4z}^+(p/2, p/2)$ is not an element of $G_1$, $\alpha_1 = C_{4z}^+L(-pa_ey)$ belongs to $G_1$. Then for $\langle a_{(m,n)} \mid a_{(m,n)^\dagger}\rangle \neq 0$, we have
\[
\langle a_{(m,n)} \mid a_{(m,n)^\dagger}\rangle = \langle \alpha_1 \cdot a_{(m,n)} \rangle \langle \alpha_1 \cdot a_{(m,n)^\dagger}\rangle
= \langle \alpha_1^2 \cdot a_{(m,n)} \rangle \langle \alpha_1^2 \cdot a_{(m,n)^\dagger}\rangle
= \langle \alpha_1^3 \cdot a_{(m,n)} \rangle \langle \alpha_1^3 \cdot a_{(m,n)^\dagger}\rangle
= \langle \alpha_1^4 \cdot a_{(m,n)} \rangle \langle \alpha_1^4 \cdot a_{(m,n)^\dagger}\rangle .
\]
(6.18)

From this we have
\[
\langle a_{(-n+p,m)} \mid a_{(-n+p,m)^\dagger}\rangle = e^{i(-\pi+2Kp)\langle a_{(m,n)} \mid a_{(m,n)^\dagger}\rangle},
\langle a_{(-m+p,-n+p)} \mid a_{(-m+p,-n+p)^\dagger}\rangle = e^{i(-\pi+2Kp(p-n))\langle a_{(-n+p,m)} \mid a_{(-n+p,m)^\dagger}\rangle},
\langle a_{(n,-m+p)} \mid a_{(n,-m+p)^\dagger}\rangle = e^{i(-\pi+2Kp(-p))\langle a_{(-m+p,-n+p)} \mid a_{(-m+p,-n+p)^\dagger}\rangle},
\langle a_{(m,n)} \mid a_{(m,n)^\dagger}\rangle = e^{i(-\pi+2Kp m)\langle a_{(m,n)} \mid a_{(m,n)^\dagger}\rangle}. \quad (6.19)
\]
Thus in the process of one rotation $(m, n) \rightarrow (-n+p, m) \rightarrow (-n+p, m-p) \rightarrow (-m+p, -n) \rightarrow (m, n)$ around the point $(l+1/2, l+1/2)$, the total change of the phase of the order parameter is 
\((-\pi+2Kp m) + (-\pi+2Kp(p-n)) + (-\pi+2Kp(p-m)) + (-\pi+2Kp n) = -2\pi.
Thus the point $(l+1/2, l+1/2)$ is a vortex line with
winding number $-1$. In a similar manner, we can easily show that other points of
corners, such as $(l + 1/2, -l - 1/2)$, are vortex lines with winding number $-1$.

Now we consider the vortex nature around a center $(l + \frac{1}{2}, 0)$ of a side of PMUC. From (2.18), a $\frac{2\pi}{2}$ rotation around the point $(l + \frac{1}{2}, 0)$ is given by

$$C_{2z}(l + \frac{1}{2}, 0) = C_{2z}T(-pe_x)$$

$$= e^{i\frac{2\pi}{3}}e^{-i\frac{\pi}{3}}C_{2z}T(-pe_x)$$

$$= e^{i\frac{\pi}{3}}C_{2z}L(-pe_x).$$

Although $C_{2z}(l + \frac{1}{2}, 0)$ is not an element of $G_1$, $\beta_1 = C_{2z}L(-pe_x)$ belongs to $G_1$. Since OPs is invariant under $\beta_1$ we have

$$\langle a_{(m,n)\downarrow}a_{(m,n)\uparrow}\rangle = \langle (\beta_1 \cdot a_{(m,n)\downarrow}) (\beta_1 \cdot a_{(m,n)\uparrow}) \rangle$$

$$= \langle (\beta_1^2 \cdot a_{(m,n)\downarrow}) (\beta_1^2 \cdot a_{(m,n)\uparrow}) \rangle.$$  

Thus we obtain

$$\langle a_{(-m+p,-n)\downarrow}a_{(-m+p,-n)\uparrow}\rangle = e^{i\pi(1-\frac{n}{p})}\langle a_{(m,n)\downarrow}a_{(m,n)\uparrow}\rangle,$$

$$\langle a_{(m,n)\downarrow}a_{(m,n)\uparrow}\rangle = e^{i\pi(1+\frac{n}{p})}\langle a_{(-m+p,-n)\downarrow}a_{(-m+p,-n)\uparrow}\rangle.$$  

Thus in the process of one rotation $(m, n) \rightarrow (-m + p, -n) \rightarrow (m, n)$ around the point $(l + 1/2, 0)$, the total change of the phase of the order parameter is $(\pi(1 - \frac{n}{p}) + \pi(1 + \frac{n}{p})) = 2\pi$. Thus the point $(l + 1/2, 0)$ is a vortex line with winding number +1. Similarly, we can see that other centers of sides of PMUC are vortex lines with winding number +1. Since a PMUC shares a corner with four other PMUCs and a side with two other PMUCs, the total winding number of a PMUC is given by $4 \times (-1) \times \frac{1}{4} + 4 \times (+1) \times \frac{1}{2} = 1$. We also see that nearest-neighbor OPs vanish at the centers of sides of PMUC. This can be seen from (6.16).

6.2. $G_2$ state

In this case the symmetry of the vortex lattice is given by the subgroup

$$G_2 = \bar{P}_2LS,$$

$$\bar{P}_2 = (e + \pi tC_{2x})C_4.$$  

The mean-field Hamiltonian and the order parameters are invariant under the action of $G_2$. From $\bar{C}_4LS$ invariance, relations (6.5), (6.6) and (6.12) hold also in this case. In the $G_2$ case, $\pi$ appears in front of $tC_{2x}, tC_{2y}, tC_{2a}$ and $tC_{2b}$. Thus the relations among the order parameters by the point symmetry become

$$\langle a_{(m,n)s}a_{(m',n')s'}\rangle = \langle a_{(-n,m)s}a_{(-n',m')s'}\rangle$$

$$= \langle a_{(-m,-n)s}a_{(-m',-n')s'}\rangle$$

$$= \langle a_{(n,-m)s}a_{(n',-m')s'}\rangle,$$

$$\langle a_{(m,n)s}a_{(m',n')s'}\rangle = \pi \langle a_{(m,-n)s}a_{(m',-n')s'}\rangle^*.$$
where \( (+ --) \) corresponds to the case \( s = s'(s \neq s') \).

\[
\begin{align*}
\langle a_{(m,0)} \downarrow a_{(m',0)} \uparrow \rangle^* &= -\langle a_{(m,0)} \downarrow a_{(m',0)} \uparrow \rangle, \\
\langle a_{(0,n)} \downarrow a_{(0,n')} \uparrow \rangle^* &= -\langle a_{(0,n)} \downarrow a_{(0,n')} \uparrow \rangle, \\
\langle a_{(m,m)} \downarrow a_{(m',m')} \uparrow \rangle^* &= -\langle a_{(m,m)} \downarrow a_{(m',m')} \uparrow \rangle, \\
\langle a_{(-m,-m)} \downarrow a_{(-m',-m')} \uparrow \rangle^* &= -\langle a_{(-m,-m)} \downarrow a_{(-m',-m')} \uparrow \rangle.
\end{align*}
\] (6.26)

Thus the above OPs must be purely imaginary. As in the case \( G_1 \), from (6.5), (6.24) and (6.25) OPs at all sites are determined from the POPs. The presence of \( C_{2z}, L(pae_x) \) and \( L(pae_y) \) symmetries leads to

\[
\begin{align*}
\langle a_{(l,0)} \downarrow a_{(l+1,0)} \uparrow \rangle &= \langle a_{(-l-1,0)} \downarrow a_{(-l,0)} \uparrow \rangle = 0, \\
\langle a_{(0,l)} \downarrow a_{(0,l+1)} \uparrow \rangle &= \langle a_{(0,-l-1)} \downarrow a_{(0,-l)} \uparrow \rangle = 0.
\end{align*}
\] (6.27)

For the case \( l = 1 \), the pattern of OPs is shown in Fig. 5. Note that all on-site order parameters \( \langle a_{(m,n)} \downarrow a_{(m,n)} \uparrow \rangle \) \( (m, n = -1, 0, 1) \) in PMUC at the origin are purely imaginary. This comes from (6.26). We see that nearest-neighbor OPs vanish at the

![Fig. 5. Schematic patterns of on-site OPs and nearest-neighbor OPs in the case \( G_2 \). The same symbols are used as in Fig. 4.](image-url)
centers of sides of PMUC. This comes from (6.27). Here we note that $G_2$ is obtained from $G_1$ as follows:

$$G_2 = (\frac{\pi}{2}) G_1 (\frac{\pi}{2})^{-1}. \quad (6.28)$$

From this, we see that OPs of the $G_2$ state can be obtained by the action of $(\frac{\pi}{2})$ on the OPs of the $G_1$ state. Figures 4 and 5 show these relations. Thus $G_2$ is physically equivalent to $G_1$.

This relation can be derived from more general considerations of co-representations. In deriving $G_i$, we consider representations of $D_4$ as usual unitary representations. However, the relevant $D_4 = C_4 + tC_{2x}C_4$ is a magnetic group with anti-unitary elements $tC_{2x}C_4$. Then we must consider co-representations. In co-representations of a magnetic group $G = H + sH$, where $H$ is a unitary subgroup and $s$ is an anti-unitary element, $D'(g)$ is equivalent to $D(g)$ if there exists a non-singular matrix $P$ such that

$$D'(h) = P^{-1}D(h)P,$$

$$D'(sh) = P^{-1}D(sh)P^*.$$

Furthermore, the basis of $D'(g)$ is given by

$$l'_i = \sum_j P_{ji}l_j. \quad (6.29)$$

where $l_i (i = 1, \cdots n)$ is a basis of $D(g)$. By substituting $A_1$ for $D(g)$ and $P = e^{i\frac{\pi}{2}}1$ (identity matrix), we see that $D'(g)$ is $A_2$ for $D_4 = C_4 + tC_{2x}C_4$, and then the basis of $A_2$ is given by multiplying $e^{i\frac{\pi}{2}}$ on the basis of $A_1$. This corresponds to the $\pi/2$ action on the OPs of $G_1$. From (6.29), if we put $P = e^{i\alpha}1$ we obtain a co-representation equivalent to $A_1$ and an equivalent OPs which is obtained by the $\alpha$ action on the OPs of $G_1$.

### 6.3. $G_3$ state

In this case the symmetry of the vortex lattice is given by the subgroup

$$G_3 = \tilde{P}_3LS,$$

$$\tilde{P}_3 = (e + tC_{2x})(e + \tilde{\pi}C_{4x}^+)(e + C_{2x}). \quad (6.31)$$

The mean-field Hamiltonian is invariant under the action of $G_3$. The difference from $G_1$ is the presence of $\tilde{\pi}$ in front of $C_{4x}^+$, $C_{4x}$, $tC_{2a}$ and $tC_{2b}$. Thus the relations among the order parameters by the space rotation symmetry are the following:

$$\langle a_{(m,n)s}a_{(m',n')s'} \rangle = -\langle a_{(-n,m)s}a_{(-m',n')s'} \rangle$$

$$= \langle a_{(-m,n)s}a_{(-m',-n')s'} \rangle$$

$$= -\langle a_{(n,-m)s}a_{(n',-m')s'} \rangle, \quad (6.32)$$

$$\langle a_{(m,n)s}a_{(m',n')s'} \rangle = \pm \langle a_{(m,-n)s}a_{(m',-n')s'} \rangle^*$$

$$= \pm \langle a_{(-m,n)s}a_{(-m',n')s'} \rangle^*$$

$$= \mp \langle a_{(n,m)s}a_{(n',m')s'} \rangle^*$$

$$= \mp \langle a_{(-n,-m)s}a_{(-n',-m')s'} \rangle^*. \quad (6.33)$$
Here \(+(-)\) corresponds to the case \(s = s' (s \neq s')\). From (6.5), (6.32) and (6.33), OPs at all sites are determined by the POPs. From (6.33) we obtain

\[
\langle a_{(m,0)}| a_{(m',0)} \rangle^* = \langle a_{(m,0)}| a_{(m',0)} \rangle, \\
\langle a_{(0,n)}| a_{(0,n')} \rangle^* = \langle a_{(0,n)}| a_{(0,n')} \rangle, \\
\langle a_{(m,m)}| a_{(m',m')} \rangle^* = -\langle a_{(m,m)}| a_{(m',m')} \rangle, \\
\langle a_{(-m,-m)}| a_{(-m',-m')} \rangle^* = -\langle a_{(-m,-m)}| a_{(-m',-m')} \rangle. 
\]  

(6.34)

The presence of \(C_{2z}, L(pa e_x)\) and \(L(pa e_y)\) symmetries leads to

\[
\langle a_{(l,0)}| a_{(l+1,0)} \rangle = \langle a_{(-l-1,0)}| a_{(-l,0)} \rangle = 0, \\
\langle a_{(0,l)}| a_{(0,l+1)} \rangle = \langle a_{(0,-l-1)}| a_{(0,-l)} \rangle = 0. 
\]  

(6.35)

Thus, as in the cases \(G_1\) and \(G_2\), we see that nearest-neighbor OPs vanish at the centers of sides of PMUC. This comes from (6.35). From \(\hat{\pi}C_{2z}\) invariance, we have

\[
\langle a_{(0,0)}| a_{(0,0)} \rangle = 0. 
\]  

(6.36)

From (6.5), the on-site OP at the center of PMUC, that is, \(\langle a_{(M_p,N_p)}| a_{(M_p,N_p)} \rangle\) is zero. Patterns of OPs in \(G_3\) are shown in Fig. 6. As we can see from Fig. 6, locations of vortex lines are the centers of PMUC and centers of sides of PMUC. As in the case of \(G_1\), applying \(\alpha_3 = e^{i\frac{\pi}{2}}C_{4z}L( - pae_y) \in G_3\) on OPs, we can see that the point

![Fig. 6. Schematic patterns of on-site OPs and nearest-neighbor OPs in the case\( G_3\). The same symbols are used as in Fig. 4.](image-url)
\((p/2, p/2)\) of a corner of PMUC is a vortex line with winding number \(+1\). Applying \(\beta_3 = e^{i\pi} C_{2z} L(-pe_x)\) on OPs, we see that the point \((p/2, 0)\) of the center of a side of PMUC is a vortex line with winding number \(-1\). Since the point \((0, 0)\) of the center of PMUC is a vortex line with winding number \((+2)\), the total winding number is \(1 \times 2 \times 1 + 4 \times 1 \times \frac{1}{4} + 4 \times (-1) \times \frac{1}{2} = 1\).

6.4. \(G_4\) state

In this case the symmetry of the vortex lattice is given by the subgroup

\[
G_4 = \tilde{P}_4 LS,
\]

\[
\tilde{P}_4 = (e + \pi tC_{2x})(e + \pi C_{4z}^{-1})(e + C_{2x}).
\]  \(6.37\)

The mean-field Hamiltonian is invariant under the action of \(G_4\). The difference from \(G_1\) is the presence of \(\pi\) in front of \(C_{4z}^{-1}, C_{4z}, tC_{2x}\) and \(tC_{2y}\). Thus the relations among order parameters by point symmetry are

\[
\langle a_{(m,n)s}a_{(m',n')s'} \rangle = -\langle a_{(-n,m)s}a_{(-n',m')s'} \rangle
\]

\[
= \langle a_{(-m,-n)s}a_{(-m',-n')s'} \rangle
\]

\[
= -\langle a_{(n,-m)s}a_{(n',-m')s'} \rangle.
\]  \(6.38\)

\[
\langle a_{(m,n)s}a_{(m',n')s'} \rangle = \mp \langle a_{(m,-n)s}a_{(-m',m')s} \rangle^*
\]

\[
= \mp \langle a_{(-m,n)s}a_{(m',n')s} \rangle^*
\]

\[
= \pm \langle a_{(n,m)s}a_{(n',m')s} \rangle^*
\]

\[
= \pm \langle a_{(-n,-m)s}a_{(-n',-m')s} \rangle^*.
\]  \(6.39\)

where \(+(-)\) corresponds to the case \(s = s'(s \neq s')\). From \(6.39\), we obtain

\[
\langle a_{(m,0)}a_{(m',0)\dagger} \rangle^* = -\langle a_{(0,m)}a_{(0,m')\dagger} \rangle,
\]

\[
\langle a_{(0,n)}a_{(0,n')\dagger} \rangle^* = -\langle a_{(n,0)}a_{(n',0')\dagger} \rangle,
\]

\[
\langle a_{(m,m)}a_{(m',m')\dagger} \rangle^* = \langle a_{(0,0)}a_{(0,0')\dagger} \rangle,
\]

\[
\langle a_{(-m,-m)}a_{(-m',-m')\dagger} \rangle^* = \langle a_{(m,m)}a_{(-m',-m')\dagger} \rangle.
\]  \(6.40\)

The presence of \(C_{2z}, L(pae_x)\) and \(L(pae_y)\) symmetries leads to

\[
\langle a_{(l,0)}a_{(l+1,0)\dagger} \rangle = \langle a_{(-l-1,0)}a_{(-l,0)\dagger} \rangle = 0,
\]

\[
\langle a_{(0,l)}a_{(0,l+1)\dagger} \rangle = \langle a_{(0,-l-1)}a_{(0,-l)\dagger} \rangle = 0.
\]  \(6.41\)

Thus, as in the cases \(G_1\) and \(G_2\), we see that nearest-neighbor OPs vanish at the centers of sides of PMUC. This comes from \(6.41\). From \(\pi C_{2z}\) invariance, we have

\[
\langle a_{(0,0)}a_{(0,0)\dagger} \rangle = 0.
\]  \(6.42\)

From \(6.5\), the on-site OP at the center of PMUC, that is, \(\langle a_{(M_P,N_p)}a_{(M_P,N_p)\dagger} \rangle\) is zero. Patterns of the \(G_4\) state are shown in Fig. 7.

As in the cases of \(G_1\) and \(G_2\), by putting \(B_1\) for \(D(g)\) and \(P = e^{i\frac{\pi}{2}}\ 1\) in \(6.29\), we can see that the representation \(B_2\) is equivalent to \(B_1\) as a co-representation and
the basis of $B_2$ is given by multiplying $e^{i\frac{\pi}{2}}$ on the basis of $B_1$. This corresponds to the $\frac{\pi}{2}$ action on the OPs of $G_3$. Thus $G_4$ is physically equivalent to $G_3$. We can see this relation by comparing Figs. 6 and 7. $G_4$ is a conjugate subgroup to $G_3$:

$$G_4 = (\pi/2)G_3(\pi/2)^{-1}. \quad (6.43)$$

6.5. $G_5$ state

In this case the symmetry of the vortex lattice is given by the subgroup

$$G_5 = \bar{P}_b L S,$$

$$\bar{P}_b = (e + tC_{2x})(e + (\pi/2)C_{4z}^+)(e + \pi C_{2z}). \quad (6.44)$$

The mean-field Hamiltonian is invariant under the action of $G_5$. Considering $\bar{C}_4 = (e, (\pi/2)C_{4z}^+ \pi C_{2z}, (\pi/2)^{-1} C_{4z}^\dagger)$ invariance and (6.12) in the $G_1$ state, we obtain

$$\langle a_{(m,n)s} a_{(m',n')s'} \rangle = e^{-i\frac{\pi}{2}} \langle a_{(-n,m)s} a_{(-n',m')s'} \rangle$$

$$= -\langle a_{(-m,-n)s} a_{(-m',-n')s'} \rangle$$

$$= e^{i\frac{\pi}{2}} \langle a_{(n,-m)s} a_{(n',-m')s'} \rangle. \quad (6.45)$$

From invariance under $tC_{2x}, \pi tC_{2y}, (\pi/2)tC_{2a}$ and $(\pi/2)^{-1}C_{2b}$ we obtain

$$\langle a_{(m,n)s} a_{(m',n')s'} \rangle = \pm \langle a_{(m,-n)s} a_{(m',-n')s'} \rangle^*$$
\[
= \mp \langle a(-m,n) \sigma a(-m',n') \sigma \rangle^* \\
= \pm e^{-i \frac{\pi}{2}} \langle a(n,m) \sigma a(n',m') \sigma \rangle^* \\
= \pm e^{i \frac{\pi}{2}} \langle a(-n,-m) \sigma a(-n',-m') \sigma \rangle^*,
\]

where \(+\(-\) corresponds to the case \(s = s'\) (\(s \neq s'\)). From (6·5), (6·45) and (6·46) the POPs determine all OPs. From (6·46) we obtain

\[
\begin{align*}
\langle a(m,0) \dagger a(m',0) \dagger \rangle^* &= \langle a(m,0) \dagger a(m',0) \rangle, \\
\langle a(0,n) \dagger a(0,n') \dagger \rangle^* &= -\langle a(0,n) \dagger a(0,n') \rangle, \\
\langle a(m,m) \dagger a(m',m') \dagger \rangle^* &= e^{i \frac{\pi}{2}} \langle a(m,m) \dagger a(m',m') \rangle, \\
\langle a(-m,m) \dagger a(-m',m') \dagger \rangle^* &= e^{-i \frac{\pi}{2}} \langle a(-m,m) \dagger a(-m',m') \rangle.
\end{align*}
\]  

(6·47)

As there is no \(C_{2z}\) in \(G_5\), (6·41) does not hold in this case. From \(\pi C_{2z}\) invariance we have

\[
\langle a(0,0) \dagger a(0,0) \dagger \rangle = 0.
\]  

(6·48)

From (6·5), on-site OP at the center of PMUC, that is, \(\langle a(M_p,N_p) \dagger a(M_p,N_p) \rangle\) is zero. Patterns of the OPs are shown in Fig. 8. From Fig. 8 we see that there is a vortex line with winding number +1 at the center of PMUC. Note that this state corresponds to an Abrikosov’s square vortex lattice.

Fig. 8. Schematic patterns of on-site OPs and nearest-neighbor OPs in the case \(G_5\). The same symbols are used as in Fig. 4.
6.6. $G_6$ state

In this case the symmetry of the vortex lattice is given by the subgroup

$$G_6 = \tilde{P}_6 LS,$$

$$\tilde{P}_6 = (e + tC_{2a})(e + (\pi/2)C_{4z}^t)(e + \tau C_{2z}).$$  

(6.49)

The mean-field Hamiltonian is invariant under the action of $G_6$. Since $G_6$ contains $\tilde{C}_4 = (e, (\pi/2)C_{4z}^t, \tau C_{2z}, (\pi/2)^{-1}C_{4z})$, we have

$$\langle a_{(m,n)s}a_{(m',n')s'} \rangle = e^{-i\frac{\pi}{2}} \langle a_{(-n,m)s}a_{(-n',m')s'} \rangle$$

$$= -\langle a_{(-m,-n)s}a_{(m',-n')s'} \rangle$$

$$= e^{i\frac{\pi}{2}} \langle a_{(n,-m)s}a_{(n',-m')s'} \rangle,$$

(6.50)

as in the case of $G_5$. From $(\pi/2)^{-1}tC_{2x}, (\pi/2)tC_{2y}, tC_{2a}$ and $\tau C_{2b}$ we obtain

$$\langle a_{(m,n)s}a_{(m',n')s'} \rangle = \pm e^{i\frac{\pi}{2}} \langle a_{(m,n)s}a_{(m',n')s'} \rangle^*$$

$$= \pm e^{-i\frac{\pi}{2}} \langle a_{(m,n)s}a_{(m',n')s'} \rangle^*$$

$$= \pm \langle a_{(n,m)s}a_{(n',m')s'} \rangle^*$$

$$= \mp \langle a_{(n,m)s}a_{(n',m')s'} \rangle^*,$$

(6.51)

where $+(−)$ corresponds to the case $s = s' (s \neq s')$. From (6.5), (6.50) and (6.51), POPs determine all OPs. From (6.51) we obtain

$$\langle a_{(m,0)\downarrow}a_{(m',0)\uparrow} \rangle^* = e^{-i\frac{\pi}{2}} \langle a_{(m,0)\downarrow}a_{(m',0)\uparrow} \rangle,$$

$$\langle a_{(0,n)\downarrow}a_{(0,n')\uparrow} \rangle^* = e^{i\frac{\pi}{2}} \langle a_{(0,n)\downarrow}a_{(0,n')\uparrow} \rangle,$$

$$\langle a_{(m,m)\downarrow}a_{(m',m')\uparrow} \rangle^* = \langle a_{(m,m)\downarrow}a_{(m',m')\uparrow} \rangle,$$

$$\langle a_{(-m,m)\downarrow}a_{(-m',m')\uparrow} \rangle^* = -\langle a_{(-m,m)\downarrow}a_{(-m',m')\uparrow} \rangle.$$

(6.52)

As there is no $C_{2z}$ in $G_6$, (6.41) does not hold. From $\tau C_{2z}$ invariance, we have

$$\langle a_{(0,0)\downarrow}a_{(0,0)\uparrow} \rangle = 0.$$  

(6.53)

From (6.5), the on-site OP at the center of PMUC, that is, $\langle a_{(M_p,N_p)\downarrow}a_{(M_p,N_p)\uparrow} \rangle$ is zero. Patterns of the OPs of $G_6$ state are shown in Fig. 9. Since $G_6$ is conjugate to $G_5$,

$$G_6 = (3\pi/4)G_5(3\pi/4)^{-1}.$$  

(6.54)

The OPs of the $G_6$ state are obtained from the action $3\pi/4$ on the OPs of the $G_5$ state. We can see this relation by comparing Figs. 8 and 9. Thus $G_6$ is physically equivalent to $G_5$. The $G_6$ state corresponds to an Abrikosov’s square vortex lattice state. $G_6$ can be considered as the state obtained from representation $E$ by putting $P = e^{i\frac{3\pi}{4}}1$ in (6.29).

Until now we have considered only symmorphic subgroups of $G_6$. In these symmorphic subgroups, the centers of point rotations are restricted to lattice points.
Fig. 9. Schematic patterns of on-site OPs and nearest-neighbor OPs in the case $G_6$. The same symbols are used as in Fig. 4.

$(m, n) (m, n : \text{integer})$. Now we consider non-symmorphic subgroups of $G_0$, in which the center of point rotations is the center of a plaquette or a link of the crystal lattice.

First, we consider a vortex lattice state whose center of rotations is $(1/2,1/2)$. From (2.18) $\sim$ (2.20), we can obtain the non-symmorphic invariance group $G_i'$ corresponding to $G_i$ as follows:

$$
G_i' = \gamma G_i \gamma^{-1}
= \gamma \tilde{P}_i \gamma^{-1} \gamma L \gamma^{-1} \gamma S \gamma^{-1}
= \tilde{P}_i' L' S
$$

(6.55)

where

$$
\gamma = T((1/2)a e_x + (1/2)a e_y),
\tilde{P}_i = \gamma \tilde{P}_i \gamma^{-1},
L' = \gamma L \gamma^{-1}.
$$

(6.56)

Using (2.18) $\sim$ (2.20), we obtain expressions for generators of $G_1'$ as follows:

$$
(C_{4z}^+)' = \gamma C_{4z}^+ \gamma^{-1}
= e^{-i\frac{\pi}{2}} C_{4z}^+ T(-ae_y),
$$
\[(tC_{2x})' = \gamma tC_{2x} \gamma^{-1} = e^{i\frac{K}{2}} tC_{2x} T(-ae_y),\]
\[(L(pae_x))' = \gamma L(pae_x)\gamma^{-1} = e^{-iK_P} L(pae_x),\]
\[(L(pae_y))' = \gamma L(pae_y)\gamma^{-1} = e^{iK_P} L(pae_y).\]

(6·57)

Note that these generators belong to \(G_0\). Then \(G'_i\) is a subgroup of \(G_0\). Rotational elements such as \((C^+_{4z})'\) and \((tC_{2x})'\) are rotations accompanying fractional translation of \(L\). We call such a subgroup a non-symmorphic subgroup. Using the same method we obtain expressions for generators of other \(G'_i(i \neq 1)\). Since \(\gamma\) does not belong to \(G_0\), \(G'_i\) is not physically equivalent to \(G_i\).

Now we consider vortex lattice states whose center of point rotations is \((1/2, 0)\). Using (2·18) \sim (2·20) we have for \(\eta = T((1/2a)e_x)\)

\[(C^+_{4z})'' \equiv C^+_{4z}(1/2, 0) = \eta C^+_{4z} \eta^{-1} = e^{-iK'/4} C^+_{4z} T\left(-\frac{1}{2}ae_x - \frac{1}{2}ae_y\right),\]
\[(C_{2x})'' \equiv C_{2x}(1/2, 0) = \eta C_{2x} \eta^{-1} = C_{2x} T(-ae_x),\]
\[(tC_{2x})'' \equiv tC_{2x}(1/2, 0) = \eta tC_{2x} \eta^{-1} = tC_{2x},\]
\[(tC_{2a})'' \equiv tC_{2a}(1/2, 0) = \eta tC_{2a} \eta^{-1} = e^{-iK/4} tC_{2a} T\left(-\frac{1}{2}ae_x + \frac{1}{2}ae_y\right),\]
\[(L(pae_x))'' \equiv \eta L(pae_x)\eta^{-1} = L(pae_x),\]
\[(L(pae_y))'' \equiv \eta L(pae_y)\eta^{-1} = e^{iK_P} L(pae_y).\] (6·58)

Then \((C^+_{4z})''\), \((C^-_{4z})''\), \((tC_{2a})''\) and \((tC_{2a})''\) do not belong to \(G_0\). Thus in this case the non-symmorphic subgroup \(G''_1\) corresponding to \(G_1\) is expressed by

\[G''_1 = (e + tC_{2x})''(e + (C_{2x})'')L''S = (e + tC_{2x})(e + C_{2x} T(-ae_x))L''S,\] (6·59)

where \(L''\) is a group generated by \(L(pae_x)\) and \(e^{iK_P} L(pae_y)\). Using the same method, we have \(G''_i\) corresponding to \(G_i(i = 3, 5)\):

\[G''_3 = (e + tC_{2x})(e + C_{2x} T(-ae_x))L''S,\]
\[G''_5 = (e + tC_{2x})(e + \bar{\eta} C_{2x} T(-ae_x))L''S.\] (6·60)

Note that \(G''_3 = G''_7\).

Now we consider vortex states whose centers of point rotations are \((m + \frac{1}{2}, n + \frac{1}{2})\).

Since

\[T\left(\left(m + \frac{1}{2}\right)ae_x + \left(n + \frac{1}{2}\right)ae_y\right) = e^{-i\frac{K}{2}(m-n)}T(mae_x + nae_y)\gamma,\] (6·61)
we have

\[ G_{i,(m+1/2,n+1/2)} \equiv T \left( \left( m + \frac{1}{2} \right) a_{e_x} + \left( n + \frac{1}{2} \right) a_{e_y} \right) G_i T \left( \left( m + \frac{1}{2} \right) a_{e_x} + \left( n + \frac{1}{2} \right) a_{e_y} \right)^{-1} \]

\[ = T(m a_{e_x} + n a_{e_y}) \gamma G_i \gamma^{-1} T(m a_{e_x} + n a_{e_y})^{-1} \]

\[ = T(m a_{e_x} + n a_{e_y}) G_i T(m a_{e_x} + n a_{e_y})^{-1}. \]  

(6.62)

As \( T(m a_{e_x} + n a_{e_y}) \) belongs to \( G_0 \), \( G_{i,(m+1/2,n+1/2)} \) is physically equivalent to \( G_i' \).

Next we consider vortex states whose center of point rotations are \( (m + \frac{1}{2}, n) \).

Since

\[ T \left( \left( m + \frac{1}{2} \right) a_{e_x} + n a_{e_y} \right) = e^{i K_n T(m a_{e_x} + n a_{e_y}) \eta}, \]  

(6.63)

we have

\[ G_{i,(m+1/2,n)} \equiv T \left( \left( m + \frac{1}{2} \right) a_{e_x} + n a_{e_y} \right) G_i T \left( \left( m + \frac{1}{2} \right) a_{e_x} + n a_{e_y} \right)^{-1} \cup G_0 \]

\[ = T(m a_{e_x} + n a_{e_y})(\eta G_i \eta^{-1} \cup G_0) T(m a_{e_x} + n a_{e_y})^{-1} \]

\[ = T(m a_{e_x} + n a_{e_y}) G_i'' T(m a_{e_x} + n a_{e_y})^{-1}. \]  

(6.64)

Then \( G_{i,(m+1/2,n)} \) is physically equivalent to \( G_i'' \).

In the following we consider vortex states whose centers of point rotations are \( (m, n + \frac{1}{2}) \). Since

\[ T \left( m a_{e_x} + \left( n + \frac{1}{2} \right) a_{e_y} \right) = e^{-i K_n T(m a_{e_x} + n a_{e_y}) T \left( \frac{1}{2} a_{e_y} \right),} \]

\[ T \left( \frac{1}{2} a_{e_y} \right) = C_{4z}^+ T \left( \frac{1}{2} a_{e_x} \right) C_{4z}^-, \]  

(6.65)

we have

\[ G_{i,(m,n+1/2)} \equiv T \left( m a_{e_x} + \left( n + \frac{1}{2} \right) a_{e_y} \right) G_i T \left( m a_{e_x} + \left( n + \frac{1}{2} \right) a_{e_y} \right)^{-1} \cup G_0 \]

\[ = T(m a_{e_x} + n a_{e_y}) C_{4z}^+ \eta C_{4z}^- G_i C_{4z}^- \eta^{-1} C_{4z}^+ T(m a_{e_x} + n a_{e_y})^{-1} \cup G_0. \]  

(6.66)

Since \( C_{4z}^- G_i C_{4z}^+ = G_i \), we have

\[ G_{i,(m,n+1/2)} = T(m a_{e_x} + n a_{e_y}) C_{4z}^+ (\eta G_i \eta^{-1} \cup G_0) C_{4z}^- T(m a_{e_x} + n a_{e_y})^{-1} \]

\[ = \{T(m a_{e_x} + n a_{e_y}) C_{4z}^+ \} G_i'' \{T(m a_{e_x} + n a_{e_y}) C_{4z}^- \}^{-1}. \]  

(6.67)

As \( \{T(m a_{e_x} + n a_{e_y}) C_{4z}^+ \} \) belongs to \( G_0 \), \( G_{i,(m,n+1/2)} \) is physically equivalent to \( G_i'' \).

Thus we obtain three types of non-symmetric vortex states \( G_i'(i = 1, 3, 5) \) for which the centers of rotations are at \( (m + \frac{1}{2}, n + \frac{1}{2}) \) and two types of non-symmetric vortex states \( G_i''(i = 1, 5) \) for which the centers of rotations are at \( (m + \frac{1}{2}, n) \) or \( (m, n + \frac{1}{2}) \). Note that \( L' \) and \( L'' \) are group isomorphic to \( L \).
§7. On the local symmetric order parameters

In §6 we classified square vortex states according to point rotation symmetry. Now we consider a system which becomes a d-wave SC state in the absence of a magnetic field. Then we will show that it is inappropriate to describe the vortex lattice state of the system in a magnetic field as a state having only d-wave OPs, the amplitude and the phase of which vary spatially.

In several papers, \(^{9,10,17}\) s-wave or d-wave order parameters on site \((m, n)\) are considered. These are defined by

\[
S(m, n) = \frac{1}{4} \left\{ \langle a_{(m,n)} | a_{(m+1,n)} \rangle + \langle a_{(m,n)} | a_{(m-1,n)} \rangle \\
+ \langle a_{(m,n)} | a_{(m,n+1)} \rangle + \langle a_{(m,n)} | a_{(m,n-1)} \rangle \right\},
\]

\[
D(m, n) = \frac{1}{4} \left\{ \langle a_{(m,n)} | a_{(m+1,n)} \rangle + \langle a_{(m,n)} | a_{(m-1,n)} \rangle \\
- \langle a_{(m,n)} | a_{(m,n+1)} \rangle - \langle a_{(m,n)} | a_{(m,n-1)} \rangle \right\}. \tag{7.1}
\]

In this section we show that these two types of order parameters necessarily coexist in the vortex lattice state by the \(L\) symmetry. We define p-wave order parameters on site \((m, n)\) by

\[
P_x(m, n) = \frac{1}{2} \left\{ \langle a_{(m,n)} | a_{(m+1,n)} \rangle - \langle a_{(m,n)} | a_{(m-1,n)} \rangle \right\},
\]

\[
P_y(m, n) = \frac{1}{2} \left\{ \langle a_{(m,n)} | a_{(m,n+1)} \rangle - \langle a_{(m,n)} | a_{(m,n-1)} \rangle \right\}. \tag{7.2}
\]

From these definitions we obtain

\[
\langle a_{(m,n)} | a_{(m+1,n)} \rangle = S(m, n) + D(m, n) + P_x(m, n),
\]

\[
\langle a_{(m,n)} | a_{(m-1,n)} \rangle = S(m, n) + D(m, n) - P_x(m, n),
\]

\[
\langle a_{(m,n)} | a_{(m,n+1)} \rangle = S(m, n) - D(m, n) + P_y(m, n),
\]

\[
\langle a_{(m,n)} | a_{(m,n-1)} \rangle = S(m, n) - D(m, n) - P_y(m, n). \tag{7.3}
\]

Using (6.5) we have

\[
\langle a_{(m+M_p,n+Np)} | a_{(m+1+M_p,n+1+Np)} \rangle = e^{i \frac{\pi}{2} (Mn-Nm)} e^{i \frac{\pi}{2} N} \langle a_{(m,n)} | a_{(m+1,n)} \rangle,
\]

\[
\langle a_{(m+M_p,n+Np)} | a_{(m-1+M_p,n-1+Np)} \rangle = e^{i \frac{\pi}{2} (Mn-Nm)} e^{i \frac{\pi}{2} N} \langle a_{(m,n)} | a_{(m+1,n)} \rangle,
\]

\[
\langle a_{(m+M_p,n+Np)} | a_{(m+M_p,n+1+Np)} \rangle = e^{i \frac{\pi}{2} (Mn-Nm)} e^{i \frac{\pi}{2} M} \langle a_{(m,n)} | a_{(m,n+1)} \rangle,
\]

\[
\langle a_{(m+M_p,n+Np)} | a_{(m+M_p,n-1+Np)} \rangle = e^{i \frac{\pi}{2} (Mn-Nm)} e^{i \frac{\pi}{2} M} \langle a_{(m,n)} | a_{(m,n-1)} \rangle. \tag{7.4}
\]
Then we have from (7.1) \(\sim\) (7.3)

\[
S(m + Mp, n + Np) = \frac{1}{4} \left\{ \langle a_{(m+Mp,n+Np)} | a_{(m+1+Mp,n+Np)} \rangle + \langle a_{(m+Mp,n+Np)} | a_{(m-1+Mp,n+Np)} \rangle + \langle a_{(m+Mp,n+Np)} | a_{(m+Mp,n+1+Np)} \rangle + \langle a_{(m+Mp,n+Np)} | a_{(m+Mp,n-1+Np)} \rangle \right\}
\]

\[
= \frac{1}{4} e^{i\pi(MN+M+N)} e^{i\frac{\pi}{2p}(Mn-Nm)}
\times \left\{ e^{-i\frac{\pi}{2p}N} \langle a_{(m,n)} | a_{(m+1,n)} \rangle + e^{i\frac{\pi}{2p}N} \langle a_{(m,n)} | a_{(m-1,n)} \rangle + e^{i\frac{\pi}{2p}M} \langle a_{(m,n)} | a_{(m,n+1)} \rangle + e^{-i\frac{\pi}{2p}M} \langle a_{(m,n)} | a_{(m,n-1)} \rangle \right\}
\]

\[
= \frac{1}{4} e^{i\pi(MN+M+N)} e^{i\frac{\pi}{2p}(Mn-Nm)}
\times \left\{ e^{-i\frac{\pi}{2p}N} (S(m,n) + D(m,n) + P_x(m,n)) + e^{i\frac{\pi}{2p}N} (S(m,n) + D(m,n) - P_x(m,n)) + e^{i\frac{\pi}{2p}M} (S(m,n) - D(m,n) + P_y(m,n)) + e^{-i\frac{\pi}{2p}M} (S(m,n) - D(m,n) - P_y(m,n)) \right\}
\]

\[
= \frac{1}{2} e^{i\pi(MN+M+N)} e^{i\frac{\pi}{2p}(Mn-Nm)}
\times \left\{ \left( \cos \left( \frac{\pi N}{2p} \right) + \cos \left( \frac{\pi M}{2p} \right) \right) S(m,n) + \left( \cos \left( \frac{\pi N}{2p} \right) - \cos \left( \frac{\pi M}{2p} \right) \right) D(m,n) - i \sin \left( \frac{\pi N}{2p} \right) P_x(m,n) + i \sin \left( \frac{\pi M}{2p} \right) P_y(m,n) \right\}.
\] (7.5)

In similar manners we obtain relations of OPs at different magnetic unit cells:

\[
D(m + Mp, n + Np) = \frac{1}{2} e^{i\pi(MN+M+N)} e^{i\frac{\pi}{2p}(Mn-Nm)}
\times \left\{ \left( \cos \left( \frac{\pi N}{2p} \right) - \cos \left( \frac{\pi M}{2p} \right) \right) S(m,n) + \left( \cos \left( \frac{\pi N}{2p} \right) + \cos \left( \frac{\pi M}{2p} \right) \right) D(m,n) + i \sin \left( \frac{\pi N}{2p} \right) P_x(m,n) - i \sin \left( \frac{\pi M}{2p} \right) P_y(m,n) \right\},
\]

\[
P_x(m + Mp, n + Np) = e^{i\pi(MN+M+N)} e^{i\frac{\pi}{2p}(Mn-Nm)}
\times \left\{ -i \sin \left( \frac{\pi N}{2p} \right) (S(m,n) + D(m,n)) \right\}.
\]
\[ P_y(m + Mp, n + Np) = e^{i\pi(MN + M + N)}e^{i\pi_p(Mn - Nm)} \times \left\{ i \sin \left( \frac{\pi M}{2p} \right) (S(m, n) - D(m, n)) + \cos \left( \frac{\pi M}{2p} \right) P_y(m, n) \right\}. \] (7-6)

Thus we see that even if there is only d-wave order parameters \( D(m, n) \) in a magnetic unit cell there necessarily appear other types of OPs \((S(m, n), P_x(m, n), P_y(m, n))\) in other magnetic unit cells. Then it is incorrect to describe the vortex lattice state by assigning a d-wave order parameter \( D(m, n) \) at each site. Contrast this case with states in no magnetic field, in which only one type of OP among s-wave, d-wave and p-wave exists in a superconducting state. Then we cannot classify superconducting states in magnetic field into s-wave, d-wave or p-wave states. The above argument also works for non-symmmorphic states with \( L' \) or \( L'' \).

§8. Irreducible bases of the invariance magnetic translation group \( L \) on the space of Fermion operators

In §§4 and 6 we have determined the invariance group \( G_i \) of the vortex lattice state. The mean-field Hamiltonian \( H_m \) is invariant under \( G_i \). Then the quasiparticles (Bogolons) corresponding to \( H_m \) are expressed by irreducible bases of \( G_i \). For this purpose we will obtain an irreducible projective representation (IPR) of the invariance magnetic translation group \( L = \{ L(pae_x), L(pae_y) \} \) of the vortex lattice state in the space of the Fermion operator. From §4,

\[ L(Mpae_x + Npae_y)a^\dagger_{(m,n)s} = e^{i\frac{\pi}{2}(MN + M + N)}e^{i\frac{\pi}{2p}(Mn - Nm)}a^\dagger_{(m + Mp, n + Np)s}. \] (8-1)

From this we have

\[ L(pae_x)L(pae_y)a^\dagger_{(m,n)s} = L(pae_x + pae_y)a^\dagger_{(m,n)s}, \]

\[ L(pae_y)L(pae_x)a^\dagger_{(m,n)s} = -L(pae_x + pae_y)a^\dagger_{(m,n)s}. \] (8-2)

We assume that following boundary conditions hold:

\[ X^{2N_0}a^\dagger_{(m,n)s} = e^{i2\pi(p+n)N_1}a^\dagger_{(m+2N_0p, n)s} = a^\dagger_{(m,n)s}, \]

\[ Y^{2N_0}a^\dagger_{(m,n)s} = e^{i2\pi(p-m)N_1}a^\dagger_{(m,n+2N_0p)s} = a^\dagger_{(m,n)s}, \] (8-3)

where

\[ X \equiv L(pae_x), \]

\[ Y \equiv L(pae_y), \] (8-4)

and we have assumed that

\[ a^\dagger_{(m+2N_0p, n)s} \equiv a^\dagger_{(m,n)s}, \]

\[ a^\dagger_{(m,n+2N_0p)s} \equiv a^\dagger_{(m,n)s}, \] (8-5)
with \( N_0 = 2pN_1, N_1 = \text{integer} \). Thus in the space of the Fermion creation and annihilation operator \( W_F = \{ a_{(m,n)s}^\dagger, a_{(m,n)s} \} \), we have the following defining relations of generators \( X \) and \( Y \) of \( L \):

\[
X^{2N_0} = e, \quad Y^{2N_0} = e, \quad XY = -YX,
\] (8.6)

where \( e \) denotes the identity of \( L \). Here we use the IPR matrix of Brown's paper, \(^{14}\)

\[
D^k(X) = e^{-ik_xpa} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
D^k(Y) = e^{-ik_ypa} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\] (8.7)

where

\[
k_x = \frac{2\pi n_x}{2paN_0}, \quad (n_x = 0, \ldots, N_0 - 1)
\]

\[
k_y = \frac{2\pi n_y}{2paN_0}, \quad (n_y = 0, \ldots, N_0 - 1)
\] (8.8)

Thus there are \( N_0^2 \) IPRs of two dimension. In the limit \( N_0 \to \infty \), \( k = (k_x, k_y) \) may be taken as \(-\frac{\pi}{2pa} \leq k_x, k_y \leq \frac{\pi}{2pa} \). We call this region of \( k \) the magnetic Brillouin zone (MBZ). We use the method of projection operators \(^{16}\) of projective representation in order to obtain irreducible bases of IPR (8.7) on \( W_F \). Applying the projection operators with \( D_{11} \) and \( D_{21} \) on \( a_{(i,j)s}^{\dagger}(-l \leq i, j \leq l) \), we obtain

\[
b_{1k(i,j)s}^{\dagger} = \frac{1}{\sqrt{2}} \left( a_{k(i,j)s}^{\dagger} + e^{i\frac{\pi}{2}p} a_{k(i+p,j)s}^{\dagger} \right),
\]

\[
b_{2k(i,j)s}^{\dagger} = \frac{1}{\sqrt{2}} \left( e^{i\frac{\pi}{2}p} a_{k(i,j+p)s}^{\dagger} + e^{i\frac{\pi}{2}p} a_{k(i+p,j+p)s}^{\dagger} \right). \] (8.9)

Applying the projection operators with \( D_{12} \) and \( D_{22} \) on \( a_{(i,j)s}^{\dagger}(-l \leq i, j \leq l) \), we obtain

\[
c_{1k(i,j)s}^{\dagger} = \frac{1}{\sqrt{2}} \left( e^{i\frac{\pi}{2}p} a_{k(i,j+p)s}^{\dagger} - e^{i\frac{\pi}{2}p} a_{k(i+p,j+p)s}^{\dagger} \right),
\]

\[
c_{2k(i,j)s}^{\dagger} = \frac{1}{\sqrt{2}} \left( a_{k(i,j)s}^{\dagger} - e^{i\frac{\pi}{2}p} a_{k(i+p,j)s}^{\dagger} \right), \] (8.10)

where

\[
a_{k(i,j)s}^{\dagger} = \frac{1}{L} \sum_{M,N} e^{ik_x(2Mp+i)a+ik_y(2Np+j)a} L(2Mpae_x)L(2Npae_y) a_{(i,j)s}^{\dagger}
\]

\[
= \frac{1}{L} \sum_{M,N} e^{ik_x(2Mp+i)a+ik_y(2Np+j)a} e^{i\pi M(1+\frac{1}{p})} e^{i\pi N(1+\frac{1}{p})} a_{(2Mp+i,2Np+j)s}^{\dagger}
\]

\[
= \frac{1}{N_0} \sum_{M,N} e^{ik_x(2Mp+i)a+ik_y(2Np+j)a} e^{i\pi (M+N)} e^{i\frac{\pi}{p} (Mj-Ni)} a_{(2Mp+i,2Np+j)s}^{\dagger}. \] (8.11)
Here \((i, j)\) is a lattice point in the conventional magnetic unit cell (MUC) consisting \((2p \times 2p)\) sites at the origin such that \(-l \leq i, j \leq l+1\), \(M\) and \(N\) run over all MUCs. The inverse transform of \((8-11)\) is given by

\[
a_k^{\dagger}(2Mp+i, 2Np+j)_s = e^{-i\frac{\pi}{p}(M+N)}e^{-i\frac{\pi}{p}(Mj-Ni)}
\times \frac{1}{N_0} \sum_k e^{-ik_x(2Mp+i)a-ik_y(2Np+j)a}a_k^{\dagger}(i,j)_s. \quad (8-12)
\]

Here \(a_k^{\dagger}(i,j)_s\) satisfies the Bloch theorem:

\[
L(2pae_x)a_k^{\dagger}(i,j)_s = e^{-2ik_xpa}a_k^{\dagger}(i,j)_s, \\
L(2pae_y)a_k^{\dagger}(i,j)_s = e^{-2ik_ypa}a_k^{\dagger}(i,j)_s. \quad (8-13)
\]

Also, \(b_qk(i,j)_s\) and \(c_qk(i,j)_s\) for \(q = 1, 2\) satisfy the magnetic Bloch theorem:

\[
L(pae_x)b_{1k(i,j)}^{\dagger}_s = e^{-ik_xpa}b_{1k(i,j)}^{\dagger}_s, \quad L(pae_x)c_{1k(i,j)}^{\dagger}_s = e^{-ik_xpa}c_{1k(i,j)}^{\dagger}_s, \\
L(pae_x)b_{2k(i,j)}^{\dagger}_s = e^{-ik_xpa}b_{2k(i,j)}^{\dagger}_s, \quad L(pae_x)c_{2k(i,j)}^{\dagger}_s = e^{-ik_xpa}c_{2k(i,j)}^{\dagger}_s, \\
L(pae_y)b_{1k(i,j)}^{\dagger}_s = e^{-ik_ypa}b_{1k(i,j)}^{\dagger}_s, \quad L(pae_y)c_{1k(i,j)}^{\dagger}_s = e^{-ik_ypa}c_{1k(i,j)}^{\dagger}_s, \\
L(pae_y)b_{2k(i,j)}^{\dagger}_s = e^{-ik_ypa}b_{2k(i,j)}^{\dagger}_s, \quad L(pae_y)c_{2k(i,j)}^{\dagger}_s = e^{-ik_ypa}c_{2k(i,j)}^{\dagger}_s. \quad (8-14)
\]

Thus \(\{b_{1k(i,j)}^{\dagger}, b_{2k(i,j)}^{\dagger}\}\) and \(\{c_{1k(i,j)}^{\dagger}, c_{2k(i,j)}^{\dagger}\}\) for \(-l \leq i, j \leq l\) are irreducible bases of IPR \((8-7)\), respectively. By \((8-9)\) and \((8-10)\) \(a_k^{\dagger}(i,j)_s\) can be expressed in terms of \(b_{qk(i,j)}^{\dagger}\) and \(c_{qk(i,j)}^{\dagger}\) for \(q = 1, 2\) and \(-l \leq i, j \leq l\):

\[
a_k^{\dagger}(i,j)_s = \frac{1}{\sqrt{2}}(b_{1k(i,j)}^{\dagger}_s + c_{2k(i,j)}^{\dagger}_s), \\
a_k^{\dagger}(i+p,j)_s = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{2}}e^{-i\frac{\pi}{2p}j}(b_{1k(i,j)}^{\dagger}_s - c_{2k(i,j)}^{\dagger}_s), \\
a_k^{\dagger}(i,j+p)_s = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{2}i}(b_{1k(i,j)}^{\dagger}_s + c_{1k(i,j)}^{\dagger}_s), \\
a_k^{\dagger}(i+p,j+p)_s = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{2}i}(b_{2k(i,j)}^{\dagger}_s - c_{1k(i,j)}^{\dagger}_s). \quad (8-15)
\]

Thus from \((8-12)\) and \((8-15)\) we can express \(a_k^{\dagger}(2Mp+i, 2Np+j)_s\) (for \(-l \leq i, j \leq l+1\)) in terms of \(b_{qk(i,j)}^{\dagger}\) and \(c_{qk(i,j)}^{\dagger}\) (for \(q = 1, 2\) and \(-l \leq i, j \leq l\)).

§9. Standard form of the mean-field Hamiltonian for a square vortex lattice

In this section we obtain the standard form of the mean-field Hamiltonian for a square vortex lattice in terms of irreducible bases of the invariance magnetic trans-
lational group \( \mathbf{L} \). In general, the mean-field Hamiltonian for the singlet superconductivity derived from the total Hamiltonian (2.7) is given by

\[
H_m = H_x^t + H_y^t + H_\mu + H_d^U + H_d^V + H_b^V + H_S^U + H_S^V, \tag{9.1}
\]

where

\[
H_x^t = -t \sum_{(m,n)s} e^{iK_m a^\dagger_{(m,n)s} a_{(m+1,n)s} + \text{h.c.}},
\]

\[
H_y^t = -t \sum_{(m,n)s} e^{-iK_m a^\dagger_{(m,n)s} a_{(m,n+1)s} + \text{h.c.}},
\]

\[
H_\mu = -\mu \sum_{(m,n)s} a^\dagger_{(m,n)s} a_{(m,n)s},
\]

\[
H_d^U = U \sum_{(m,n)s} \langle a^\dagger_{(m,n)\uparrow} a_{(m,n)\downarrow} \rangle a^\dagger_{(m,n)s} a_{(m,n)s},
\]

\[
H_d^V = V \sum_{(m,n)s} \left\{ \langle a^\dagger_{(m+1,n)\uparrow} a_{(m,n)\downarrow} \rangle + \langle a^\dagger_{(m,n-1)\uparrow} a_{(m,n)\downarrow} \rangle a^\dagger_{(m,n)s} a_{(m,n)s} \right\},
\]

\[
H_b^V = -V \sum_{(m,n)s} \left\{ \langle a^\dagger_{(m,n+1)\uparrow} a_{(m,n)\downarrow} \rangle \right\} a^\dagger_{(m,n)s} a_{(m,n+1)s} + \text{h.c.},
\]

\[
H_S^U = U \sum_{(m,n)} \langle a_{(m,n)\uparrow} a_{(m,n)\downarrow} \rangle \left\{ a^\dagger_{(m,n)\uparrow} a^\dagger_{(m,n)\downarrow} - a^\dagger_{(m,n)\downarrow} a^\dagger_{(m,n)\uparrow} \right\} + \text{h.c.},
\]

\[
H_S^V = V \sum_{(m,n)} \langle a_{(m,n)\uparrow} a_{(m+1,n)\downarrow} \rangle \left\{ a^\dagger_{(m,n)\uparrow} a^\dagger_{(m+1,n)\downarrow} - a^\dagger_{(m,n)\downarrow} a^\dagger_{(m+1,n)\uparrow} \right\} + \text{h.c.},
\]

\[
+ V \sum_{(m,n)} \langle a_{(m,n)\downarrow} a_{(m,n+1)\uparrow} \rangle \left\{ a^\dagger_{(m,n)\downarrow} a^\dagger_{(m,n+1)\uparrow} - a^\dagger_{(m,n)\uparrow} a^\dagger_{(m,n+1)\downarrow} \right\} + \text{h.c.} \tag{9.2}
\]

Using (8.12) and (8.15) we can express each part of \( H_m \) by the irreducible bases \( b_{qk(i,j)s}^\dagger \) and \( c_{qk(i,j)s}^\dagger \) (\( q = 1, 2 \)) as follows:

\[
H_x^t = -t \sum_{k_s} \sum_{-l \leq i < l} \sum_{-l \leq j \leq l} e^{iK_j} e^{i k x a} \sum_q \left\{ b_{qk(i,j)s}^\dagger b_{qk(i+1,j)s} + c_{qk(i,j)s}^\dagger c_{qk(i+1,j)s} \right\} + \text{h.c.},
\]

\[
- t \sum_{k_s} \sum_{-l \leq i \leq l} e^{iK_j} e^{i k x a} e^{i \frac{\pi}{2} p} \sum_q \left\{ b_{qk(l,j)s}^\dagger b_{qk(-l,j)s} - c_{qk(l,j)s}^\dagger c_{qk(-l,j)s} \right\} + \text{h.c.},
\]

\[
H_y^t = -t \sum_{k_s} \sum_{-l \leq i \leq l} \sum_{-l \leq j \leq l} e^{-iK_j} e^{i k y a} \sum_q \left\{ b_{qk(i,j)s}^\dagger b_{qk(i,j+1)s} + c_{qk(i,j)s}^\dagger c_{qk(i,j+1)s} \right\} + \text{h.c.},
\]

\[
- t \sum_{k_s} \sum_{-l \leq i \leq l} e^{-iK_j} e^{i k y a} e^{-i \frac{\pi}{2} p} \sum_q \left\{ b_{qk(l,i)s}^\dagger c_{qk(-l,i)s} + c_{qk(l,i)s}^\dagger b_{qk(-l,i)s} \right\} + \text{h.c.},
\]
\[ H_\mu = -\mu \sum_{k_s} \sum_{-l \leq i, j \leq l} \{ b_{qk(i,j)}^\dagger b_{qk(i,j)} + c_{qk(i,j)}^\dagger c_{qk(i,j)} \}, \]
\[ H_d^U = U \sum_{k_s} \sum_{-l \leq i, j \leq l} \langle a_{(i,j)}^\dagger a_{(i,j)} \rangle \sum_q \{ b_{qk(i,j)}^\dagger b_{qk(i,j)} + c_{qk(i,j)}^\dagger c_{qk(i,j)} \}, \]
\[ H_d^V = V \sum_{k_s} \sum_{-l \leq i, j \leq l} \{ \langle a_{(i+1,j)}^\dagger a_{(i+1,j)} \rangle + \langle a_{(i-1,j)}^\dagger a_{(i-1,j)} \rangle \}
+ \langle a_{(i,j+1)}^\dagger a_{(i,j+1)} \rangle + \langle a_{(i,j-1)}^\dagger a_{(i,j-1)} \rangle \}
\times \sum_q \{ b_{qk(i,j)}^\dagger b_{qk(i,j)} + c_{qk(i,j)}^\dagger c_{qk(i,j)} \}, \]
\[ H_b^V = -V \sum_{k_s} \sum_{-l \leq i < l - l \leq l} \sum_{-l \leq i \leq l} e^{ikx} a_{(i,j)}^\dagger a_{(i+1,j)} \}
\times \sum_q \{ b_{qk(i,j)}^\dagger b_{qk(i+1,j)} + c_{qk(i,j)}^\dagger c_{qk(i+1,j)} \} + \text{h.c.} \]
\[ -V \sum_{k_s} \sum_{-l \leq i \leq l} e^{ikx} e^{i \frac{x_i}{2} - i \frac{x_l}{2}} (a_{(i,l)}^\dagger a_{(i+l-1,j)}) \}
\times \sum_q \{ b_{qk(i,j)}^\dagger b_{qk(-l,j)} - c_{qk(i,j)}^\dagger c_{qk(-l,j)} \} + \text{h.c.} \]
\[ -V \sum_{k_s} \sum_{-l \leq i \leq l} e^{ikx} a_{(i,j)}^\dagger a_{(i,j+1)} \}
\times \sum_q \{ b_{qk(i,j)}^\dagger b_{qk(i,j+1)} + c_{qk(i,j)}^\dagger c_{qk(i,j+1)} \} + \text{h.c.} \]
\[ -V \sum_{k_s} \sum_{-l \leq i \leq l} e^{ikx} e^{i \frac{x_i}{2} - i \frac{x_l}{2}} (a_{(i,l)}^\dagger a_{(i+l-1,j)}) \}
\times \sum_q \{ b_{qk(i,l)}^\dagger c_{qk(i,-l)} + c_{qk(i,l)}^\dagger b_{qk(i,-l)} \} + \text{h.c.}, \quad (9.3) \]
\[ H_S^U = U \sum_{k_s} \sum_{-l \leq i, j \leq l} \langle a_{(i,j)}^\dagger a_{(i,j)} \rangle \]
\times \sum_q \{ b_{qk(i,j)}^\dagger b_{q-k(i,j)} + a_{qk(i,j)}^\dagger c_{q-k(i,j)} \}
\times \sum_q \{ b_{qk(i,j)}^\dagger b_{q-k(i,j)} + a_{qk(i,j)}^\dagger c_{q-k(i,j)} \} + \text{h.c.}, \]
\[ H_S^V = V \sum_{k_s} \sum_{-l \leq i, j \leq l} \sum_{-l \leq i \leq l} e^{ikx} a_{(i,j)}^\dagger a_{(i+1,j)} \}
\times \sum_q \{ (b_{qk(i,j)}^\dagger b_{q-k(i+1,j)} + c_{qk(i,j)}^\dagger c_{q-k(i+1,j)}) \}
\times \sum_q \{ (b_{qk(i,j)}^\dagger b_{q-k(i+1,j)} + c_{qk(i,j)}^\dagger c_{q-k(i+1,j)}) \} + \text{h.c.} \]
\[ + V \sum_{k_s} \sum_{-l \leq i \leq l} e^{ikx} e^{i \frac{x_i}{2} - i \frac{x_l}{2}} (a_{(i,l)}^\dagger a_{(l+1,j)}) \}
\times \sum_q \{ (b_{qk(i,j)}^\dagger b_{q-k(l,j)} - c_{qk(i,j)}^\dagger c_{q-k(l,j)}) \} \]
\[
\begin{align*}
-\left( b_{qk(i,j)}^\dagger b_{q-k(l,j)}^\dagger - c_{qk(i,j)}^\dagger c_{q-k(l,j)}^\dagger \right) + \text{h.c.} \\
+V \sum_k \sum_{-l \leq i \leq l} \sum_{-l \leq j < l} e^{ik_ya} \langle a_{(i,j)}^\dagger a_{(i,j+1)} \rangle \\
\times \sum_q \left\{ (b_{qk(i,j)}^\dagger b_{q-k(i,j+1)}^\dagger + c_{qk(i,j)}^\dagger c_{q-k(i,j+1)}^\dagger) \\
- (b_{qk(i,j)}^\dagger b_{q-k(i,j+1)}^\dagger + c_{qk(i,j)}^\dagger c_{q-k(i,j+1)}^\dagger) \right\} + \text{h.c.} \\
+V \sum_k \sum_{-l \leq i \leq l} e^{ik_ya} e^{-i\frac{\pi}{2}} e^{i\frac{\pi}{2} p} \langle a_{(i,l)}^\dagger a_{(i,l+1)} \rangle \\
\times \sum_q \left\{ (b_{qk(i,l)}^\dagger c_{q-k(i,-l)}^\dagger + c_{qk(i,l)}^\dagger b_{q-k(i,-l)}^\dagger) \\
- (b_{qk(i,l)}^\dagger c_{q-k(i,-l)}^\dagger + c_{qk(i,l)}^\dagger b_{q-k(i,-l)}^\dagger) \right\} + \text{h.c.}
\end{align*}
\]

(9.4)

In deriving these expressions we used the following relations:

\[
\begin{align*}
\langle a_{(i+1+M_p,j+N_p)}^\dagger a_{(i+1+M_p,j+N_p)} \rangle &= e^{\mp i\frac{\pi N}{2p}} \langle a_{(i,j)}^\dagger a_{(i+1,j)} \rangle, \\
\langle a_{(i+1+M_p,j+N_p)}^\dagger a_{(i+1+M_p,j+N_p)} \rangle &= e^{\pm i\frac{\pi N}{2p}} \langle a_{(i,j)}^\dagger a_{(i+1,j)} \rangle, \\
\langle a_{(i+M_p,j+N_p)}^\dagger a_{(i+M_p,j+N_p)} \rangle &= e^{i\pi (M+N)} e^{i\pi MN} e^{i\frac{\pi}{2p} (M-j-N_i)} \langle a_{(i,j)}^\dagger a_{(i,j)} \rangle, \\
\langle a_{(i+1+M_p,j+N_p)}^\dagger a_{(i+1+M_p,j+N_p)} \rangle &= e^{i\pi (M+N)} e^{i\pi MN} e^{-i\frac{\pi N}{2p}} e^{i\frac{\pi}{2p} (M-j-N_i)} \langle a_{(i,j)}^\dagger a_{(i+1,j)} \rangle, \\
\langle a_{(i+M_p,j+N_p)}^\dagger a_{(i+M_p,j+N_p)} \rangle &= e^{i\pi (M+N)} e^{i\pi MN} e^{i\frac{\pi}{2p} (M-j-N_i)} \langle a_{(i,j)}^\dagger a_{(i+1,j)} \rangle, \\
\langle a_{(i+1+M_p,j+N_p)}^\dagger a_{(i+1+M_p,j+N_p)} \rangle &= e^{i\pi (M+N)} e^{i\pi MN} e^{i\frac{\pi}{2p} (M-j-N_i)} \langle a_{(i,j)}^\dagger a_{(i+1,j)} \rangle,
\end{align*}
\]

(9.5)

which are derived from (6.5). Now let us express \( \langle a_{(i,j)}^\dagger a_{(i,j)} \rangle \) in terms of \( \langle b_{qk(i,j)}^\dagger b_{qk(i,j)} \rangle \) and \( \langle c_{qk(i,j)}^\dagger c_{qk(i,j)} \rangle \). Using (8.12) we obtain

\[
\begin{align*}
\langle a_{(i,j)}^\dagger a_{(i,j)} \rangle &= \frac{1}{N_0} \sum_k e^{-ik_xa_{(i,j)}^\dagger a_{k(i,j)}}, \\
\langle a_{(i,j)}^\dagger a_{(i,j)} \rangle &= \frac{1}{N_0} \sum_k e^{ik_xa_{(i,j)}^\dagger a_{k(i,j)}}.
\end{align*}
\]

(9.6)

Then for \(-l \leq i, j \leq l\)

\[
\begin{align*}
\langle a_{(i,j)}^\dagger a_{(i,j)} \rangle &= \frac{1}{N_0} \sum_k \sum_{k'} e^{i(k'_x a_{(i,j)}^\dagger a_{k(i,j)}^\dagger (a_{k'(i,j)}^\dagger a_{k(i,j)}))} \\
&= \frac{1}{N_0} \sum_k \langle a_{(i,j)}^\dagger a_{k(i,j)}^\dagger a_{k(i,j)} \rangle \\
&= \frac{1}{2} \frac{1}{N_0} \sum_k \left\{ \langle b_{k(i,j)}^\dagger b_{k(i,j)}^\dagger c_{2k(i,j)}^\dagger c_{2k(i,j)} \rangle + \langle c_{2k(i,j)}^\dagger c_{2k(i,j)}^\dagger b_{k(i,j)}^\dagger b_{k(i,j)} \rangle \right\},
\end{align*}
\]

(9.7)

where we have used (8.15) and \( \langle a_{(i,j)}^\dagger a_{k(i,j)}^\dagger a_{k'(i,j)} \rangle = 0 \) for \( k' \neq k \), derived from (8.13) as well as \( L(2pa e_x) \) and \( L(2pa e_y) \) invariance. From \( L(pa e_y) b_{1k(i,j)}^\dagger = e^{-ik_y a} b_{2k(i,j)}^\dagger \)
and \( L(pa_{ey})c_{1k(i,j)s}^{\dagger} \) we obtain
\[
\langle b_{2k(i,j)s}^{\dagger} b_{2k(i,j)s} \rangle = \langle b_{1k(i,j)s}^{\dagger} b_{1k(i,j)s} \rangle, \\
\langle c_{2k(i,j)s}^{\dagger} c_{2k(i,j)s} \rangle = \langle c_{1k(i,j)s}^{\dagger} c_{1k(i,j)s} \rangle. 
\]
(9-8)
Thus we have
\[
\langle a_{(i,j)s}^{\dagger} a_{(i,j)s} \rangle = \frac{1}{2 N_0} \sum_k \{ \langle b_{1k(i,j)s}^{\dagger} b_{1k(i,j)s} \rangle + \langle c_{1k(i,j)s}^{\dagger} c_{1k(i,j)s} \rangle \}. 
\]
(9-9)
Using similar methods, we obtain the following expressions (\(-l \leq i \leq l, -l \leq j \leq l\) if unspecified):
\[
\langle a_{(i,j)s}^{\dagger} a_{(i,j)s} \rangle = \frac{1}{2 N_0} \sum_k \{ \langle b_{1k(i,j)s}^{\dagger} b_{1k(i,j)s} \rangle + \langle c_{1k(i,j)s}^{\dagger} c_{1k(i,j)s} \rangle \}, \\
\langle a_{(i,j)s}^{\dagger} a_{(i+1,j)s} \rangle = \frac{1}{2 N_0} \sum_k e^{ikx_{a}} \{ \langle b_{1k(i,j)s}^{\dagger} b_{1k(i,j)s} \rangle \} \\
- \langle c_{1k(i+1,j)s}^{\dagger} c_{1k(i+1,j)s} \rangle \text{ for } -l \leq i < l, \\
\langle a_{(i,j)s}^{\dagger} a_{(l+1,j)s} \rangle = \frac{1}{2 N_0} \sum_k e^{i \frac{\pi}{2} \frac{x}{p}} e^{i \frac{\pi}{2} \frac{y}{p}} e^{ikx_{a}} \{ \langle b_{1k(i,j)s}^{\dagger} b_{1k(l,j)s} \rangle \} \\
- \langle c_{1k(l,j)s}^{\dagger} c_{1k(l,j)s} \rangle, \\
\langle a_{(i,j)s}^{\dagger} a_{(i,j+1)s} \rangle = \frac{1}{2 N_0} \sum_k e^{iky_{a}} \{ \langle b_{1k(i,j)s}^{\dagger} b_{1k(i,j+1)s} \rangle \} \\
+ \langle c_{1k(i,j+1)s}^{\dagger} c_{1k(i,j+1)s} \rangle \text{ for } -l \leq j < l, \\
\langle a_{(i,l)s}^{\dagger} a_{(i,l+1)s} \rangle = \frac{1}{2 N_0} \sum_k e^{i \frac{\pi}{2} \frac{x}{p}} e^{-i \frac{\pi}{2} \frac{y}{p}} e^{iky_{a}} \{ \langle b_{1k(i,l)s}^{\dagger} c_{1k(i,l)s} \rangle \} \\
+ \langle c_{1k(i,l)s}^{\dagger} b_{1k(i,l)s} \rangle, \\
\langle a_{(i,j)\dagger} a_{(i,j)\dagger} \rangle = \frac{1}{2 N_0} \sum_k \{ \langle b_{1-k(i,j)} b_{1k(i,j)} \rangle \} \\
+ \langle c_{1-k(i,j)} c_{1k(i,j)} \rangle, \\
\langle a_{(i,j)\dagger} a_{(i+1,j)\dagger} \rangle = \frac{1}{2 N_0} \sum_k e^{ikx_{a}} \{ \langle b_{1-k(i,j)} b_{1k(i+1,j)} \rangle \} \\
+ \langle c_{1-k(i+1,j)} c_{1k(i+1,j)} \rangle \text{ for } -l \leq i < l, \\
\langle a_{(i,j)\dagger} a_{(i,j+1)\dagger} \rangle = \frac{1}{2 N_0} \sum_k e^{iky_{a}} \{ \langle b_{1-k(i,j)} b_{1k(i,j+1)} \rangle \} \\
+ \langle c_{1-k(i,j+1)} c_{1k(i,j+1)} \rangle \text{ for } -l \leq j < l, \\
\langle a_{(l,j)\dagger} a_{(l+1,j)\dagger} \rangle = \frac{1}{2 N_0} e^{\frac{i}{2} x} e^{\frac{i}{2} y} \sum_k e^{ikx_{a}} \{ \langle b_{1-k(l,j)} b_{1k(l,j)} \rangle \} \\
- \langle c_{1-k(l,j)} c_{1k(l,j)} \rangle \};
\[ \langle a_{(i,l)}| a_{(i,l+1)} \rangle = \frac{1}{2N_0} e^{\frac{i\pi}{2p}} e^{-\frac{i\pi}{2p}} \sum_k e^{ik_\alpha} \left\{ \langle b_{1-k(i,l)}| c_{1k(i,l)} \rangle + \langle c_{1-k(i,l)}| b_{1k(i,l)} \rangle \right\}. \]  

(9-10)

Thus matrix elements for \( q = 1 \) and \( q = 2 \) of \( H_m \) are the same and are expressed in terms of only the first components of the irreducible bases, \( b_{1k(i,j)} \) and \( c_{1k(i,j)} \). Note that in \( H_m \), \( (i,j) \) run over PMUC (that is, \( p^2 \) sites) and there are two kinds of Fermions, \( b^\dagger \) and \( c^\dagger \). Then, diagonalization of \( H_m \) by the Bogoliubov transformation (the size of its unitary matrix is \( 4p^2 \times 4p^2 \)) yields the same quasiparticle band energies such that \( E^1_{\alpha}(k) = E^2_{\alpha}(k) (\alpha = 1, \ldots, 2p^2) \). Thus the bands of quasiparticles are always doubly degenerate. The creation operators of quasiparticles with quasiparticle energies \( E^1_{\alpha}(k) = E^2_{\alpha}(k) \) and \( E^1_{\alpha}(-k) = E^2_{\alpha}(-k) \) are expressed as follows:

\[ \gamma^\dagger_{q \alpha k} = \sum_{(i,j)} \left\{ v^b_{\alpha,i,j}(k) b^\dagger_{qk(i,j)} + v^c_{\alpha,i,j}(k) c^\dagger_{qk(i,j)} \right\} + \sum_{(i,j)} \left\{ v^b_{\alpha,i,j}(k) b_{q-k(i,j)} + v^c_{\alpha,i,j}(k) c_{q-k(i,j)} \right\}, \]

\[ \gamma^\dagger_{q \alpha -k} = \sum_{(i,j)} \left\{ v^b_{\alpha,i,j}(-k) b^\dagger_{q-k(i,j)} + v^c_{\alpha,i,j}(-k) c^\dagger_{q-k(i,j)} \right\} - \sum_{(i,j)} \left\{ v^b_{\alpha,i,j}(-k) b_{qk(i,j)} + v^c_{\alpha,i,j}(-k) c_{qk(i,j)} \right\}. \]

(9-11)

From (8.14) we can easily see that the \( \gamma^\dagger_{q \alpha k} \) \((q = 1, 2)\) transform same as the \( b_{qk(i,j)} \) for \( L(pae_x) \) and \( L(pae_y) \). We note that the magnetic Brillouine zone \( \{-k_x, k_y \leq \frac{\pi}{2pa} \} \) is a half of that in previous papers, while the size of Bogoliubov transformation is the same as that in previous papers.\(^{10,11}\) The reduction of the magnetic Brillouine zone by a half is compensated for by the double degeneracy of the band of quasiparticles.

§10. Summary and discussion

We have given a group theoretical classification of singlet superconducting states (vortex lattice states) in a uniform magnetic field of the extended Hubbard model including an on-site attraction \((U < 0)\) or nearest-neighbor attraction \((V < 0)\). Using symmetries of magnetic translation and tetragonal rotation of the system we have obtained invariance groups of vortex lattice states. When the magnetic flux \( \phi \) through a unit cell of the crystal lattice is

\[ \phi = \phi_0 \frac{2J}{p_1p'_1 - p_2p'_2}, \quad (J = \text{integer}) \]

or

\[ \phi = \phi_0 \frac{J}{p_1p'_2 - p_2p'_1}, \quad (J = \text{odd integer}) \]

(10-1)

for integer \( p_1, p_2, p'_1, p'_2 \), a vortex lattice with basis vector \( \mathbf{l}_1 = p_1ae_x + p_2ae_y \) and \( \mathbf{l}_2 = p'_1ae_x + p'_2ae_y \) is possible. We have studied in detail the case \( \mathbf{l}_1 = pae_x, \mathbf{l}_2 = pae_y \).
and $\phi = \phi_0 \frac{1}{p^2}$. In this case we obtained three types of symmorphic vortex lattice states $G_i = \mathbf{P}_i \mathbf{L} (i = 1, 3, 5)$, for which the center of point rotations is at a lattice point. In these states, locations and winding numbers of vortex lines are determined by the symmetry $G_i$.

From these three symmorphic states, we have derived three non-symmorphic vortex lattice states $G'_i (i = 1, 3, 5)$, for which the center of point rotations is at the center of a plaquette of crystal lattice, and two non-symmorphic vortex lattice states $G''_i (i = 1, 5)$, for which the center of point rotations is at the center of a link of the crystal lattice.

We have shown that local symmetric order parameters, such as $d$-wave, $s$-wave and $p$-wave order parameters, are meaningless, since three types ($d$, $s$, $p$-wave) of order parameters always coexist by the symmetry of $\mathbf{L}$.

An irreducible projective representation of $\mathbf{L}$ in the space of Fermion operators was obtained. The mean-field Hamiltonian $H_m$ is expressed in terms of irreducible bases of $\mathbf{L}$. We showed the following:

1. The quasi particles form bands in magnetic Brillouin zone $\{-\frac{\pi}{2p_0} \leq k_x, k_y \leq \frac{\pi}{2p_0}\}$, which is a half of the one used in previous papers. \cite{10}, \cite{11}

2. All bands are at least doubly degenerate.

3. The dimension of $H_m$ is $4p^2$.

4. The reduction of the magnetic Brillouin zone is compensated for by a double degeneracy of bands.

A double degeneracy of the bands comes from the two-dimensional irreducible projective representation of the invariance magnetic translation group $\mathbf{L}$ in the case $\phi/\phi_0 = 1/p^2$. This two-dimensionality also occurs for $\mathbf{L}'_i, \mathbf{L}''_i$ and more general vortex lattices with basis vectors $\mathbf{l}_1 = p_1 a \mathbf{e}_x + p_2 a \mathbf{e}_y$ and $\mathbf{l}_2 = p'_1 a \mathbf{e}_x + p'_2 a \mathbf{e}_y$ such that

$$\phi \phi_0 = \frac{J}{p_1 p'_2 - p_2 p'_1}.$$ \hspace{1cm} (10.2)

where $J$ is an odd integer. Finally, we note that another phase rearrangement from (4.8) is possible:

$$L^z(\mathbf{M}_1 + \mathbf{N}_2) a_{(m,n)s}^\dagger = e^{i\frac{\pi}{2}MN} T(\mathbf{M}_1 + \mathbf{N}_2) a_{(m,n)s}^\dagger.$$ 

We can construct a similar argument using $L^z$ instead of $\mathbf{L}$. The choice of $\mathbf{L}$ or $L^z$ should be determined by numerical calculations of the total free energy. In order to obtain the phase diagram of SC states in magnetic fields, we must calculate the total free energy of the above group theoretically possible states. We note that a similar analysis works also for a triplet superconducting states.

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References

1) I. Luk'yanchuk, J. de Phys. 11 (1991), 1155.
19) The Reference to this paper suggested the possibility of the vortex lattice states having vortex lines at centers of plaquettes or centers of links of the crystal lattice. These correspond to our non-symorphic vortex lattice states. He also suggested of the locations and winding numbers of vortex line of $G_1$ and $G_3$ states.