The Problem of Time in the Quantum Theory of Black Holes

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We discuss the problem of time in spherically symmetric pure Einstein gravity with the cosmological term by using an exact solution to the Wheeler-DeWitt equation. A positive definite inner product is defined, based on the momentum constraint rather than the Hamiltonian constraint. A natural notion of time is introduced via the Heisenberg equation. This notion enables one to reproduce the time-time component of the classical metric. Non-Hermiticity of the Hamiltonian is essential in the definition of time.

The canonical theory of the general relativity reveals the complete constraint structure of the theory. Replacing the canonical momenta by functional derivatives with respect to the corresponding fields, we obtain the Wheeler-DeWitt (WDW) equation from the classical constraints. A long-standing problem in quantum cosmology is how to interpret the wave functional in the WDW theory.

A major problem results from the fact that the total Hamiltonians of any models with general coordinate invariance vanish. Thus any observables in such a theory are absolutely static with respect to the time coordinate. This is referred as the “problem of time” (see, for example, Ref. 5). In the classical theory a surface term can be added to the Hamiltonian (see, for example, Ref. 6), but this violates the general coordinate invariance which must be preserved in the full quantum theory. This prevents us from comparing the classical theory with the full quantum theory. In the semiclassical approximation, the problem may be viewed as superficial, since one can define a reference clock of a semiclassical nature, and the WDW equation reduces to a Schrödinger-type equation of first order with respect to differentiation by the clock variable, while the latter is related to the coordinate time via a classical equation of motion. However, the genuine problem in the full quantum theory is that one should derive, in some limits, the classical metric from the quantum theory itself.

Another problem of the WDW theory, referred to as the “Hilbert space problem”, concerns the hyperbolic signature of the DeWitt metric in the superspace, which may lead to negative probability, as in the case of the Klein-Gordon wave equation. This has also been treated in a semiclassical approximation and in minisuperspace approaches, and interesting results and concepts such as the third quantization have been proposed.

In the present paper we consider the above two problems in spherically symmetric Einstein gravity with the cosmological term using an exact solution to the WDW equation. An exact solution in the effectively two-dimensional gravity theories was first obtained in 10) for Jackiw-Teitelboim gravity, and in using similar method in

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for spherically symmetric gravity without matter fields. Exact WDW solutions
for the CGHS model,\textsuperscript{13} a two-dimensional gravity with a dilaton, were obtained in
Ref. 14) in the case with conformal matter fields, and in Ref. 15) in the case without
matter fields. In order to analyze the effect of Hawking radiation one needs matter
fields. Unfortunately no exact solution has been found in the spherically symmetric
theory with matter fields due to their coupling to the scale factor (dilaton field). Here
we restrict ourselves to the two problems mentioned above and apply the solution
obtained in 14) to the spherically symmetric theory, neglecting the matter fields.

Let us start with the spherically symmetric world line element written as
\begin{equation}
 ds^2 = -g_{\mu\nu}(t,r)dx^\mu dx^\nu - e^{-2\varphi(t,r)}(d\theta^2 + \sin^2 \theta d\phi),
\end{equation}
where \(e^{-2\varphi}\) is a scale factor. It turns out that the entire calculation is drastically
simplified if one introduces the zweibein \(e_{\mu\alpha}\) defined by \(g_{\mu\nu} = e_{\mu\alpha}e_{\nu\beta}\) (we choose
the signature of the flat metric as \(\eta_{11} = -\eta_{00} = 1\)). The Einstein action with the
cosmological term is
\begin{equation}
 S = \frac{1}{16\pi G} \int d^4x (4) \epsilon (4) R - 2\lambda
 = \frac{1}{4G} \int d^2x e \left[ e^{-2\varphi}(R + 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 2\lambda) + 2 \right],
\end{equation}
where \(G\) is the gravitational constant and the superscript 4 designates the four-
dimensional quantities. The technical advantage of using the zweibein variable is manifested in the canonical theory in the local Lorentz gauge defined by\textsuperscript{17}
\begin{equation}
 e_{\mu\alpha} = \begin{pmatrix} \alpha & \beta \rho \\ 0 & e^\rho \end{pmatrix},
\end{equation}
where \(\alpha\) and \(\beta\) are the shift and the lapse functions, respectively, and \(\rho\) is another
dynamical variable.

The canonical theory of the spherically symmetric Einstein gravity was obtained
long ago,\textsuperscript{16} and the hamiltonian (with the cosmological term) is expressed as
\begin{equation}
 H = \int dr (\beta \Phi_1 + \alpha \Phi_2 + f_1 \Pi_\beta + f_2 \Pi_\alpha),
\end{equation}
where \(f_1 = \beta\) and \(f_2 = \alpha\) are not determined by the Legendre transformation and,
therefore, are arbitrary functions of the canonical variables. \(\Phi_1\) and \(\Phi_2\) do not depend
on \(\alpha\) and \(\beta\), and are expressed in terms of the canonical momenta, \(\Pi_\rho, \Pi_\varphi, \Pi_\alpha\) and
\(\Pi_\beta\), conjugates of \(\rho, \varphi, \alpha\) and \(\beta\), respectively, as
\begin{equation}
 \Phi_1 = e^{-2\rho} \left[ \left( \rho' - \frac{\partial}{\partial r} \right) \Pi_\rho + \varphi' \Pi_\varphi \right],
\end{equation}
\begin{equation}
 \Phi_2 = \frac{1}{2} e^{-\rho} \left[ G e^{2\rho} (\Pi_\rho + 2\Pi_\varphi) \Pi_\rho + \frac{e^{-2\varphi}}{G} R_1 \right],
\end{equation}
\begin{equation}
 R_1 = 3\varphi'^2 - 2\varphi^{[2]} + \lambda e^{2\rho} - e^{2\rho+2\varphi}.
\end{equation}
Here the one-dimensional covariant derivatives\(^{14}\) are defined by \(\varphi^0 \equiv \varphi, \varphi^{[n+1]} \equiv (\partial / \partial r - n \rho') \varphi^n (n \geq 1)\) (primes denote derivatives with respect to \(r\)). \(R_1\) is the one-dimensional intrinsic curvature.

The WDW equation then becomes

\[
\left[ \left( \rho'(r) - \frac{\partial}{\partial \rho(r)} \right) \frac{\delta}{\delta \rho(r)} + \varphi'(r) \frac{\delta}{\delta \varphi(r)} \right] \Psi[\rho, \varphi] = 0, \tag{8}
\]

\[
\left[ \left( \frac{\delta}{\delta \rho(r)} + 2 \frac{\delta}{\delta \varphi(r)} \right) \frac{\delta}{\delta \rho(r)} - \frac{e^{-4\varphi}}{G^2} R_1 \right] \Psi[\rho, \varphi] = 0. \tag{9}
\]

Because of the constraints, \(\Pi_\alpha = \Pi_\beta = 0\), the wave functional \(\Psi\) cannot depend on \(\alpha\) and \(\beta\). Since the WDW equation contains second order functional differential operators at the same point, it requires some regularization. In this paper we adopt the DeWitt prescription,\(^3\) \(\delta(\pi - r) = 0\). Furthermore, we set \(\delta^{(n)}(r - r) = 0 (n \geq 1)\).

The consistency of this regularization scheme will not be discussed in this paper, and it should be considered as one of the rules of the game. There have been some attempts to derive the quantum effects of matter fields, in a semiclassical expansion for gravity, by setting \(\delta(0) \neq 0\).\(^{18}\) This seems to be consistent only if one makes modifications of the WDW equation.\(^{19}\) Another motive for our approach comes from the fact that our quantum theory is in fact an effective theory of the full four-dimensional theory rather than a genuine two-dimensional theory, where some Schwinger terms are unavoidable.

The key idea to solve the WDW equations is to introduce possible scalar and scalar densities out of \(\rho\) and \(\varphi\) in the one-dimensional sense.\(^{14}\) That is, we have two scalar densities, \(\varphi'\) and \(e^{-\rho}\), and one scalar \(\varphi\). Instead, it turns out to be convenient to use two scalars, \(\varphi\) and \(X \equiv e^{-\rho} \varphi'\), and one scalar density \(\varphi'\). Since the momentum constraint Eq. (8) amounts to general covariance in the radial dimension, we are tempted to make an ansatz for the wave functional \(\Psi\) as

\[
\Psi[\rho, \varphi] = \exp \frac{1}{G} \int dr e^{-2\varphi} \varphi' W(X, \varphi), \tag{10}
\]

where \(W\) is a complex valued function of \(X\) and \(\varphi\), which is to be determined. Then the momentum constraints Eq. (8) reduce to the requirement that \(W\) does not depend explicitly on \(r\). The Hamiltonian constraint Eq. (9) reduces to

\[
(c_{\varphi} - 3c - \lambda + e^{2\varphi}) \varphi' + cX \varphi' = 0, \tag{11}
\]

where \(c = X^2 + X^4W_\chi^2\), and \(c_{\varphi} = \partial c / \partial \varphi\), \(c_X = \partial c / \partial X\). For Eq. (11) to hold for an arbitrary function \(\varphi\), both coefficients of \(\varphi'\) and \(X\) must vanish. We see \(c\) is a function of only \(\varphi\) because \(c_X = 0\). Integrating \(c = X^2 + X^4W_\chi^2\) with respect to \(X\), and putting the coefficient of \(\varphi'\) in Eq. (11) to zero, we obtain the solution. So far we have used the variable \(\rho\), but in general \(e^{2\rho}\) becomes negative. So we define \(\gamma = e^{2\rho}\), which ranges from \(-\infty\) to \(+\infty\). The solution is then expressed as

\[
W = \eta^{1/2} + i \ln \gamma^{-1/2}(\eta^{1/2} - i) + i \ln(-\varphi') + f(\varphi), \tag{12}
\]
where
\begin{equation}
\eta = \frac{\gamma}{\gamma_c} - 1,
\end{equation}
and \( \kappa \) is an arbitrary complex number with dimension of mass. The multivalued terms in Eq. (12) represent one of the branches in the complex plane, and we assume \( \varphi' < 0 \) for simplicity. The last term, \( f(\varphi) \), in Eq. (12) is an arbitrary function of \( \varphi \), contributing to \( \Psi \) only at the boundary and playing the role of suppressing possible divergences. The wave functional is stationary, both for \( \gamma \) and \( \varphi \), at the function \( \gamma = \gamma_c \), which coincides with \( g_{11} \) in the Schwarzschild coordinate if one chooses \( e^{-2\varphi} = r^2 \). The above solution satisfies such a boundary condition that if \( \gamma(r) \) goes to \( \infty \) for a fixed value of \( r \) (except at the horizon) the wave functional vanishes, since the Hilbert space is assumed to be composed of functionals on the space of all well-behaved (continuous) functions of \( r \).

The inner product in the Hilbert space is defined by imposing its invariance with respect to a clock variable which we choose to be \( \varphi(r) \). The ordinary definition due to DeWitt is, however, not positive definite. Alternatively, we can define the positive definite inner product as
\begin{equation}
\langle \Psi_1 | \Psi_2 \rangle = \int \prod_{r'} d\rho(r') \psi_1^* \psi_2.
\end{equation}
We can easily prove that Eq. (15) is independent of \( \varphi(r) \) due to the momentum constraint Eq. (8), by which we have
\begin{equation}
\frac{\delta}{\delta \varphi(r)} \psi_1^* \psi_2 = \frac{1}{\varphi'(r)} \left( -\rho'(r) + \frac{\partial}{\partial r} \right) \frac{\delta}{\delta \rho(r)} \psi_1^* \psi_2.
\end{equation}
According to the boundary condition imposed on \( \Psi \), the functional integral of Eq. (16) over \( \rho \) vanishes (note \( \delta'(0) = 0 \)). The above argument can be applied equally to the four-dimensional case and to a model with matter fields as well. The positive definiteness of the inner product defined by Eq. (15) allows us to interpret it as the probability amplitude.

The hamiltonian operator \( H \) defined by Eq. (4) with the replacement \( \Pi_q \rightarrow -i\delta/\delta q \) for a dynamical variable \( q \), is not Hermitian in the present definition of the inner product, which contrasts with the situations considered by other authors (see, for example, Ref. 12)). However, there may be no physical reason for the Hermiticity of the hamiltonian constraint (recall the analogy to the relativistic particle).

Let us turn to the problem of obtaining the classical counterpart to the wave functional satisfying the WDW equation. Since the functional integral of \( (\delta/\delta \rho) \Psi^* \Psi \) over \( \rho \) vanishes, we see that \( \langle \eta^{1/2}(r) \rangle = 0 \). Similarly, we see that \( \langle \eta^{1/2}(r) \eta^{1/2}(r') \rangle = 0 \) for \( r \neq r' \). In the limit \( r' \rightarrow r \) we obtain \( g_{11} = \langle \gamma \rangle = \gamma_c \). Choosing the “gauge” \( e^{-2\varphi} = r^2 \), we obtain the Schwarzschild form of \( g_{11} \),
\begin{equation}
g_{11} = \left( 1 - \frac{2\kappa G}{r} - \frac{\lambda}{3} r^2 \right)^{-1}.
\end{equation}
Thus $\kappa$ is interpreted as the mass located at origin. Following reasoning similar to that above we see $\langle \eta^{n/2} \rangle = 0$ for an arbitrary integer $n$. This implies $\langle F(\gamma) \rangle = F(\gamma_c)$ for an arbitrary function $F$. For example, the one-dimensional intrinsic curvature, $R_1 = -(3\varphi'^2 - 2\varphi^{(2)})\eta + \varphi \eta'$, has a vanishing expectation value.

The derivation of the time components of the metric is not straightforward, since the notion of “time” is implicit (or undefined) in the WDW quantization scheme. Note that a state depends on the clock, which we choose to be $\varphi(r)$, but does not depend on time. Our formulation is in analogy with ordinary quantum mechanics in the Heisenberg representation, where an operator depends on time (not the clock) but a state does not. In our formulation, however, the time dependence is implicit in the field variables, and a priori there is no criterion to define the time dependence of any quantities. This is the “problem of time” mentioned above. We assume that the time development of a quantity, $\mathcal{O}$, composed of $\rho$ and $\varphi$ and the functional derivatives with respect to them is defined by

$$\dot{\mathcal{O}} = i[H, \mathcal{O}], \quad \text{(18)}$$

For example functional derivatives with respect to the field variables become time dependent, and we denote them as $\mathcal{D}_q(r, t)$ instead of $\delta/\delta q(r)$ for $q = \phi, \rho$ (we assume $\mathcal{D}_q(r, 0) = \delta/\delta q(r)$). This definition of time is meaningful only when the hamiltonian is not Hermitian, since otherwise we have $\langle \dot{\mathcal{O}} \rangle = i\langle (\mathcal{H}\mathcal{O} - \mathcal{O}\mathcal{H}) \rangle = i\langle (\mathcal{H}^{\dagger}\mathcal{O} - \mathcal{O}\mathcal{H}) \rangle = 0$ for an arbitrary operator $\mathcal{O}$.

Since the hamiltonian depends on the lapse ($\alpha$) and the shift ($\beta$) functions, the time development of any quantity is generically gauge dependent, while the WDW pair of equations is gauge independent, because of the closure of the constraint algebra. Equation (18) may be interpreted in two ways. First it is the ordinary classical canonical equation of motion for the field variables (obtained by replacing an operator with the corresponding canonical variable). In the present formulation of quantum gravity, it should be regarded as determining orbits of the operators in the Hilbert space. The notion of the hamiltonian is redundant in genuine quantum theory, but the form of the hamiltonian is consistently determined when one wishes to compare the quantum theory with the classical theory.

Since the hamiltonian is not Hermitian, an Hermitian operator generally becomes a non-Hermitian operator by time development. We postulate that the expectation value of an operator developed from an Hermitian operator, evaluated in a physical state, should be real. This requirement restricts the gauge. In what follows we show that the above condition leads to the gauge choice corresponding to the correct classical metric $g_{00}$ in the Schwarzschild coordinates.

The commutation relations are written, in the gauge $\beta = 0$, as

$$i[H, \rho] = \alpha \varphi^{-1} e^{\rho + \varphi} \left[ - (N' + N\rho') + Ge^{\varphi} \Phi_1 \right], \quad \text{(19)}$$

$$i[H, \varphi] = -\alpha e^{\rho + \varphi} N, \quad \text{(20)}$$

where $N \equiv Ge^{-2\rho + \varphi} \frac{\delta}{\delta \rho(r)}$. This quantity satisfies

$$i[H, N] = e^{-2\rho + \varphi} (\alpha + \alpha_\rho) \Phi_2 - G^{-1} e^{-3\rho - \varphi} (\alpha' \varphi' + (\varphi'^2 - \varphi^{(2)})\alpha). \quad \text{(21)}$$
If we choose
\[ \alpha = \varphi' e^{-(\rho + \varphi)}, \]  
(22)
then the r.h.s. of Eq. (21) vanishes. In this gauge we have
\[ \rho(r, t) = \rho(r) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \left[ a_n + \sum_{k=1}^{n} b_{nk} \frac{\partial^k \rho}{\partial r^k} \right] + (\Phi), \]  
(23)
\[ \varphi(r, t) = \varphi(r) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=1}^{n} b_{nk} \frac{\partial^k \varphi}{\partial r^k}, \]  
(24)
where the coefficients \( a_n \) and \( b_{nk} \) are functions of \( N \) and its derivatives, determined iteratively as \( a_1 = -N' \), \( a_{n+1} = -(N'a_n)' \) and \( b_{11} = -N \), \( b_{n+1k} = -(b'_{nk} + b_{nk-1}N), \) \( b_{m0} = b_{mn+1} = 0 \). In Eq. (23) the term \( (\Phi) \) represents terms which vanish if they act on the physical state satisfying \( \Phi_1(1) = \Phi_2(1) = 0 \). Thus we see the expectation values of \( \rho(r, t) \) and \( \varphi(r, t) \) in the physical states are real. The gauge (22) corresponds to the correct classical value of the metric, i.e., \( g_{00} = -\langle \alpha'^2 \rangle = -\gamma_c^{-1} \) for \( e^{-2\varphi} = r^2 \). Note again that there exists no condition restricting \( \alpha \) if the hamiltonian is Hermitian.

Some comments on the Heisenberg relation, Eq. (18), are in order. There are two differences between the present situation and the ordinary transformation theory of quantum mechanics; namely, in our case, the hamiltonian vanishes in the physical Hilbert space, and the hamiltonian is not Hermitian with respect to the inner product defined by Eq. (15). From these facts the wave functional in the Schrödinger representations, denoted by \( \Psi_S \), is divided into bra and ket functionals satisfying
\[ i \frac{\partial}{\partial t} \Psi_S^{(bra)} = H^\dagger \Psi_S^{(bra)}, \]  
(25)
\[ i \frac{\partial}{\partial t} \Psi_S^{(ket)} = 0, \]  
(26)
where \( H^\dagger \) is the Hermite conjugate of the hamiltonian, the existence of which is implied by the Riesz theorem. Matrix elements of an arbitrary operator in the Schrödinger representation are the same as those in the Heisenberg representation due to Eqs. (25) and (26).

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References
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