Mean Field Potential for Non-Equilibrium Phase Space
and Its Effect on Heavy-Ion Reaction

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Mean field potentials for non-equilibrium phase space distributions are obtained from
the solution of the Bethe-Goldstone equation for two interpenetrating nuclear matters. They
are given as a function of the total density, relative momentum and asymmetry of the two
nuclear matters. The mean-field potentials become less attractive as the relative momentum
increases for finite asymmetry. For convenience in studying heavy-ion reactions in the
framework of transport theories, the density dependence of the potential is parametrized in a
Skyrme-type form. Coefficients are functions of the relative momentum and the asymmetry
between the two nuclear matters. These parametrized potentials are implemented in quantum
molecular dynamics (QMD). The total density, relative momentum and asymmetry at
a local point are evaluated in the simulation and used to calculate the potential in the equa-
tion of motion. In order to study the effect of a non-equilibrium mean-field potential in the
studies of heavy-ion reactions, simulations of $^{40}$Ca-$^{40}$Ca at $E_{\text{lab}} = 200$ and 400 MeV/u have
been performed in QMD using both non-equilibrium and equilibrium mean-field potentials.
The effect is significant even at 200 MeV and should be included in studies of heavy-ion
reactions using transport theories.

§1. Introduction

One of the main aims to study heavy-ion reactions is to obtain information about
the equation of state for nuclear matter. The most successful theoretical frameworks
for studying heavy-ion reactions are the transport theories of various names (VUU,
BUU, QMD, etc.). ¹ The equation of state is reflected as the mean-field potential in
calculations within the frameworks of these transport theories. In almost all such
studies, the mean-field potential for cold and equilibrium phase space distributions
is used. It is clear, however, that hot and non-equilibrium phase space distributions
are involved in actual heavy-ion collisions. Locally, the phase space distribution is
considered to be that of two interpenetrating nuclear matters. This can be specified
by the total density, relative momentum between the two nuclear matters and the
asymmetry of the two densities. Before the collision, the asymmetry is zero for
the entire region, and nucleons feel the potential for the equilibrium phase space
distribution, but as the two nuclei begin to overlap, the asymmetry becomes finite
in the overlap region, and the nucleons in that region feel a potential which is very
different from that of the equilibrium phase space distribution, since the relative
momentum is large. As the reaction proceeds further, the asymmetry becomes finite
for wide region, but the relative momentum decreases, and the temperature increases
due to nucleon-nucleon collisions.
In order to extract meaningful information about the equation of state from the analysis of heavy-ion collisions in the transport theories, one has to test if the use of cold and equilibrium mean-field potentials throughout the simulations is justified. The effect of using a temperature dependent mean-field potential in the analysis of heavy-ion reactions in QMD has been studied in Ref. 2. It has been shown that the temperature dependence significantly affects the heavy-ion dynamics for heavy nuclei. The temperature dependent mean-field potential has been obtained from the solution of the Bethe-Goldstone equation for a single nuclear matter. In this paper we study the remaining effect, namely the effect of the non-equilibrium mean-field potential in the studies of heavy-ion reactions.

The potential energy density for two interpenetrating nuclear matters has been calculated by solving the Bethe-Goldstone equation for two Fermi spheres in Ref. 3) and successfully used in explaining the optical potential between heavy ions. The mean-field potential for such a non-equilibrium phase space distribution can be obtained from Landau’s prescription and is parametrized in a Skyrme-type form for convenience to be used in the analysis of heavy-ion reactions in transport theories. The mean-field potential becomes less attractive as the relative momentum increases for finite asymmetry. This is due to the reduction of the attractive NN-interaction in the $^3S_1-^3D_1$ channel. This means that the analysis using equilibrium mean-field potentials would need a harder equation of state than actually is since they have to compensate for the repulsion inherent in the non-equilibrium potential.

§2. Formalism

The potential energy densities for two interpenetrating nuclear matters have been obtained in Ref. 3) by solving the Bethe-Goldstone equation for a momentum distribution with two Fermi spheres. They are functions of the total density $\rho$, relative momentum $\kappa$, and the asymmetry $\eta = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$. Detailed descriptions of how to solve the Bethe-Goldstone equation can be found in Ref. 3). They are calculated for the densities $\rho/\rho_0 = 1/16, n/4 \ (n = 1, \cdots, 10)$ at relative momenta $\kappa = 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0$ and asymmetries $\eta = n/8 \ (n = 1, \cdots, 4)$, in a total of 208 sets. As has been done in Ref. 2), we parametrize the calculated potential energy density in the form of a Skyrme interaction as

$$\Pi(\rho, \kappa, \eta) = \rho \left\{ \frac{\alpha(\kappa, \eta)}{2} \left( \frac{\rho}{\rho_0} \right)^2 + \frac{\beta(\kappa, \eta)}{\gamma(\kappa, \eta)} + \frac{\gamma(\kappa, \eta)}{\gamma(\kappa, \eta) + 1} \right\} \left[ 1 + \Delta \Pi(\rho) \right], \quad (2.1)$$

where $\Delta \Pi(\rho)$, which is taken to be independent of $\kappa$ and $\eta$, is introduced to obtain a good fit at low densities. We note that $\Delta \Pi(\rho) \simeq 0$ for $\rho \geq 1.5 \rho_0$. The details of Eq. (2.1) are given in the Appendix. The mean-field potential $U(\rho, \kappa, \eta)$ for the non-equilibrium phase space distribution specified by $\kappa$ and $\eta$ can be obtained from $\Pi(\rho, \kappa, \eta)$ by using the Landau prescription,

$$U(\rho) = \partial \Pi(\rho)/\partial \rho. \quad (2.2)$$
Using the parametrized form Eq. (2-1) for $\Pi(\rho, \kappa, \eta)$, we obtain the parametrized mean-field potential as

$$U(\rho, \kappa, \eta) = \alpha(\kappa, \eta) \left( \frac{\rho}{\rho_0} \right) + \beta(\kappa, \eta) \left( \frac{\rho}{\rho_0} \right)^{\gamma(\kappa, \eta)} + \Delta U(\rho, \kappa, \eta),$$

(2-3)

where $\Delta U$ is the term arising from the term with $\Delta \Pi$ in Eq. (2-1). The calculated and parametrized values are compared in Fig. 1 for several sets of $\kappa$ and $\eta$ to show the quality of the fitting.

Since we limit the parameters to as small a number as possible, the fit is not as good as in the case of Ref. 2), but it is sufficient for practical calculations. The mean-field potentials obtained using Eq. (2-3) are plotted in Fig. 2 for several sets of $\kappa$ and $\eta$.

It can be seen that the potential becomes less attractive and then repulsive as the relative momentum increases, and the increase is more rapid for larger $\eta$. From these results we can see that the use of equilibrium mean-field potential in the transport theories becomes less justified as the incident energy increases.
According to a standard procedure\textsuperscript{4}) to construct the potential energy to be used in QMD, we obtain

$$V = \sum_{i=1}^{A_p + A_T} \left[ \frac{\alpha(\kappa_i, \eta_i)}{2} \left( \sum_{j \neq i} \tilde{\rho}_{ij} \right) + \beta(\kappa_i, \eta_i) \gamma(\kappa_i, \eta_i) \right] \left[ 1 + \Delta II(\tilde{\rho}_{ij}) \right],$$

(2.4)

with

$$\tilde{\rho}_{ij} = \frac{1}{(4\pi L)^{3/2}} \exp[-(r_i - r_j)^2/4L],$$

(2.5)

where we have assumed that the correction term $\Delta II$ appears in the potential in the same manner as in Eq. (2.1). The relative momentum $\kappa_i$ and the asymmetry $\eta_i$ are calculated at the position of the $i$th particle, $r_i$, as

$$\kappa_i = \left| \sum_{j=1}^{A_p} p_{j} \rho_j(r_i) - \sum_{j=A_p+1}^{A_p+A_T} p_{j} \rho_j(r_i) \right| \rho_P(r_i) - \rho_T(r_i),$$

(2.6)

where

$$\rho_i(r) = \frac{1}{(2\pi L)^{3/2}} \exp[-(r - r_i)^2/2L],$$

(2.7)

$$\rho_P(r) = \sum_{i=1}^{A_p} \rho_i(r),$$

(2.8)

and

$$\eta_i = \frac{\min(\rho_P(r_i), \rho_T(r_i))}{\rho_P(r_i) + \rho_T(r_i)}.$$

(2.9)

The potential energy calculated with these $\kappa_i$ and $\eta_i$ are used in the equation of motion, given by

$$\frac{dp_i}{dt} = -\frac{\partial V}{\partial r_i},$$

(2.10)

$$\frac{dr_i}{dt} = \frac{p_i}{\sqrt{p_i^2 + m_i^2}}.$$}

(2.11)

Here we treat the $\kappa$ and $\eta$ as parameters, and their implicit dependence on $r_i$ and $p_i$ is neglected in taking the derivative. This may cause a slight breaking of the total energy conservation, which should be studied further in future calculations.

\section*{§3. Results and discussion}

In order to study the effect of the non-equilibrium potential in the study of heavy-ion reactions, we have performed simulations in QMD of the $^{40}$Ca-$^{40}$Ca reaction at
$E_{\text{lab}} = 200$ and 400 MeV/u with an impact parameter $b = 2$ fm using both non-equilibrium and equilibrium mean-field potentials. In the case of a non-equilibrium phase space distribution, it is possible to calculate the NN-cross sections to be used in the QMD simulations by solving the Bethe-Goldstone equation, as discussed in Refs. 2) and 3). It is important, however, to use the same NN-cross sections in both calculations in order to single out the effect of the non-equilibrium potential. For this reason, we use the NN-cross section parametrized by Cugnon 5) in both calculations. The results for $E_{\text{lab}} = 200$ and 400 MeV/u are shown in Figs. 3 and 4, respectively.

In Figs. 3(a) and 4(a) the maximum density attained during the reaction is plotted as a function of time. It is noted that the maximum density is slightly higher for the results with the equilibrium potential. This is due to the strong repulsion of the non-equilibrium potential. In Figs. 3(b) and 4(b), the relative momentum $\kappa$ and the asymmetry $\eta$ averaged over the central region within a radius of 2 fm are shown as functions of time. Initially, the relative momentum has a value corresponding to the incident energy, and the asymmetry is small, since there is little overlap between projectile and target. As two nuclei begin to overlap, $\eta$ increases as can be seen from

Fig. 3. Time dependence of (a) the maximum density, (b) the relative momentum $\kappa$ and asymmetry $\eta$ averaged over the central region within a radius of 2 fm, and (c) the directed transverse momentum $(p_{t}^{\text{dir}})$, at $E_{\text{lab}} = 200$ MeV/u. The solid and dashed lines represent the results with non-equilibrium and equilibrium potentials, respectively.

Fig. 4. Same as Fig. 3, but for $E_{\text{lab}} = 400$ MeV/u.
Eq. (2.9), and \( \kappa \) decreases due to \( NN \) collisions. In Figs. 5 and 6, the density, mean-field potential, relative momentum and asymmetry along the \( x \)-axis \( (y = z = 0) \) are plotted at times 9.6 fm/c and 16.8 fm/c, respectively for the case \( E_{\text{lab}} = 400 \) MeV/u.

Following convention, the incident momentum direction is taken to be the \( z \)-axis and the impact parameter direction is taken to be the \( x \)-axis. At 9.6 fm/c, when the two nuclei begin to overlap, the relative momentum is still large, and the non-equilibrium potential is considerably less attractive. At 16.8 fm/c, when the maximum density is reached, the relative momentum has decreased to 1 fm\(^{-1}\), and the non-equilibrium potential is only slightly more repulsive than the equilibrium one. We note, however, that the temperature reaches its highest value at this time, as shown in Ref. 2. In Figs. 3(c) and 4(c), the directed transverse momentum defined

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**Fig. 5.** The \( x \)-dependence of (a) the total density (solid line) and the projectile and target densities (dashed lines), (b) the relative momentum \( \kappa \) (solid line) and asymmetry \( \eta \) times 10 (dashed line), and (c) the non-equilibrium mean-field potential \( U(\rho(\tau), \kappa(\tau), \eta(\tau)) \) (solid line) and equilibrium mean-field potential \( U(\rho(\tau), 0, 0) \) (dashed line), for \( E_{\text{lab}} = 400 \) MeV/u at time 9.6 fm/c.

**Fig. 6.** Same as Fig. 5, but for time 16.8 fm/c.
by
\[ \langle p_x^{\text{dir}} \rangle = \frac{1}{A_P + A_T} \sum_{i=1}^{A_P + A_T} \text{sign}(y_i) p_{ix}, \] (3.1)
where \( y_i \) is the rapidity of the particle,
\[ y_i = \frac{1}{2} \ln \frac{E_i + p_{ix}}{E_i - p_{ix}}, \] (3.2)
is shown as a function of time. At first, \( \langle p_x^{\text{dir}} \rangle \) becomes negative due to the attraction between the two nuclei. Then it increases and becomes positive due to the repulsion caused by the high density region. Finally, it reaches a constant value after the reaction has been completed. This value is about 10% higher for the equilibrium potential. It should be noted that the flow is not directly related to the compression energy, as pointed out in Ref. 6). From these results we may conclude that the effect of the non-equilibrium mean-field potential cannot be neglected in the studies of heavy-ion reactions in transport theories. It is, of course, necessary to include both the temperature dependence and the effect of the non-equilibrium phase space distribution in the mean-field potential at the same time. Such calculations are now in progress.

By using Eq. (2.2), we have neglected the momentum dependence of the mean-field potential. It is pointed out in Ref. 1) that explicit inclusion of a momentum dependence in the mean-field potential gives rise to repulsive effects. It is a remaining problem to derive the momentum dependent mean-field potentials for non-equilibrium phase space distribution.

**Appendix**

The potential energy density \( \Pi(\rho, \kappa, \eta) \) is parametrized as
\[ \Pi(\rho, \kappa, \eta) = \Pi^0(\rho, \kappa, \eta)[1 + \Delta \Pi(\rho)], \] (A.1)
where
\[ \Pi^0(\rho, \kappa, \eta) = \rho_0 \left\{ \frac{\alpha(\kappa, \eta)}{2} \left( \frac{\rho}{\rho_0} \right)^2 + \frac{\beta(\kappa, \eta)}{\gamma(\kappa, \eta) + 1} \left( \frac{\rho}{\rho_0} \right)^{\gamma(\kappa, \eta)+1} \right\}, \] (A.2)
and
\[ \Delta \Pi(\rho) = a \frac{\rho}{\rho_0} \left( 1 + b \cos \left( \frac{c \rho}{\rho_0} \right) \right) \exp \left( -d \frac{\rho}{\rho_0} \right), \] (A.3)
with \( a = 8.39645, b = 0.151159, c = 0.000125336 \) and \( d = 12.7082 \). The \( \kappa, \eta \)-dependence of \( \alpha, \beta \) and \( \gamma \) is given by
\[ \alpha(\kappa, \eta) = \alpha_0[1 + a_1(\eta)\kappa + a_2(\eta)\kappa^2], \] (A.4)
\[ \beta(\kappa, \eta) = \beta_0[1 + b_1(\eta)\kappa + b_2(\eta)\kappa^2], \] (A.5)
\[ \gamma(\kappa, \eta) = \gamma_0[1 + c_1(\eta)\kappa + c_2(\eta)\kappa^2], \] (A.6)
with $\alpha_0 = -352.322$, $\beta_0 = 294.534$, $\gamma_0 = 1.14497$ and

\begin{align}
a_n(\eta) &= \eta \sum_{i=0}^{4} a_n^{(i)} \eta^i, \quad (n = 1, 2) \quad (A.7) \\
b_n(\eta) &= \eta \sum_{i=0}^{4} b_n^{(i)} \eta^i, \quad (n = 1, 2) \quad (A.8) \\
c_n(\eta) &= \eta \sum_{i=0}^{4} c_n^{(i)} \eta^i, \quad (n = 1, 2) \quad (A.9)
\end{align}

where $\kappa$ is in fm$^{-1}$, and the coefficients $a_n^{(i)}$, $b_n^{(i)}$ and $c_n^{(i)}$ are summarized in Table I. We note that $0 \leq \eta \leq \frac{1}{2}$ and $\alpha$, $\beta$ and $\gamma$ satisfy the condition $\alpha(0, \eta) = \alpha(\kappa, 0) = \alpha_0, \ldots$, and

$$\left. \frac{\partial \alpha}{\partial \eta} \right|_{\eta=1/2} = 0, \ldots.$$  \hspace{1cm} (A.10)

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References