Spin-3/2 Ising Model and Ashkin-Teller Model

Tsuyoshi Horiguchi and Yasushi Honda

Department of Computer and Mathematical Sciences
GSIS, Tohoku University, Sendai 980-77

(Received January 30, 1995)

We derive a relation between a general spin-3/2 Ising model and an Ising model with two sets of Ising spins. A spin-3/2 Ising model with up-down symmetry is expressed in terms of an Ashkin-Teller model and solved exactly in some cases without assuming extended Horiguchi's condition. It turns out that the system on the square lattice shows rich critical phenomena.

Ising models with spin greater than 1/2 have recently been investigated by many authors. A spin-1 Ising model, which was introduced by Blume, Emery and Griffiths,\(^1\) has been solved exactly on a honeycomb lattice by Horiguchi\(^2\) and by Wu\(^3\) when Horiguchi's condition is satisfied: Horiguchi's condition is given by \(\exp(K)\cosh J=1\), where \(J\) and \(K\) are dipolar and quadrupolar interactions constants. The results are also obtained by Rosengren and Hagkvist\(^4\) and by Kolesik and Samaj\(^5\) and extended to some cases with external field by Shanker\(^6\) and by Wu and Wu.\(^7\) The investigation of the spin-1 Ising model has been extended to the spin-3/2 Ising model by Lipowski and Suzuki\(^8\) and by Morita\(^9\) and to the spin-2 Ising model by Lipowski and Suzuki.\(^8\) In those solutions, Horiguchi's condition or its extended conditions and the honeycomb lattice play an important role. However those results show only the critical phenomena of Onsager type. It is expected that a spin-3/2 Ising model shows rich critical phenomena. In this paper, we present a spin-3/2 Ising model on the square lattice which actually shows rich critical phenomena.

First we show that a general spin-3/2 Ising model is expressed in terms of an Ising model of two sets of Ising spins of \(\pm 1\). We consider a spin-3/2 Ising model with at most \(n\)-spin interactions on a finite set \(\Lambda\) of a \(d\)-dimensional lattice, whose Hamiltonian is given by

\[-\beta H(S) = \sum_{P \subseteq \Lambda} J_{P,\rho(P)} S_{P,\rho(P)},\]

(1)

where \(\beta = 1/kT\) as usual. Here we define \(S_{P,\rho(P)}\) for a subset \(P\) of \(\Lambda\) as follows:

\[S_{P,\rho(P)} = \prod_{i \in P} S_i^{\rho_i},\]

(2)

where \(S_i\) takes on \((-1, -1/3, 1/3, 1)\) and \(\rho_i\) on \((1, 2, 3)\). \(J_{P,\rho(P)}\) is an interaction constant for spins on sites belonging to the subset \(P\); \(|P| \leq n\). We assume that the ranges of the interactions are finite: it is possible to find a number \(r\) such that \(J_{P,\rho(P)}\) is zero whenever \(\text{diam}(P) > r\). We define the free energy \(f\) of the system:

\[-\beta f = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \ln Z,\]

(3)
where
\[ Z = \sum_{\{\sigma_i\}, \{u_i\}} e^{-\beta H(S)}. \] (4)

The thermodynamic limit is taken in the sense of van Hove.\(^{10}\)

We introduce two sets \( \{\sigma_i\}, \{u_i\} \) of Ising spins of \( \pm 1 \) and express \( S_i \) in terms of \( \sigma_i \) and \( u_i \) as follows:\(^{11}\)
\[ S_i = \frac{1}{3} \sigma_i (u_i + 2). \] (5)

Thus we have from Eq. (4)
\[ Z = \sum_{\{\sigma_i\}, \{u_i\}} e^{-\beta H(\sigma, u)}, \] (6)

where
\[ -\beta H(\sigma, u) = \sum_{P \in A} J_{P, \rho(P)} U_{P, \rho(P)}(\sigma, u), \] (7)
\[ U_{P, \rho(P)}(\sigma, u) = \prod_{i \in P} \frac{1}{2} \left( \frac{\sigma_i}{3} \right)^{\rho_i} [(3^{\rho_i} - 1) u_i + (3^{\rho_i} + 1)] 
= \prod_{i \in P} \left[ \frac{1}{3} \sigma_i (u_i + 2) \delta_{\rho_i, 1} + \frac{1}{9} (4 u_i + 5) \delta_{\rho_i, 2} + \frac{1}{27} \sigma_i (13 u_i + 14) \delta_{\rho_i, 3} \right]. \] (8)

Inversely, we have from Eq. (5)
\[ \sigma_i = \frac{1}{4} S_i (13 - 9 S_i^2) \] (9)
and
\[ u_i = \frac{1}{4} (9 S_i^2 - 5). \] (10)

Then we have Eqs. (1), (2) and (4) from Eqs. (6)~(9). In this way, the spin-3/2 Ising model is equivalent to an Ising model with two sets of Ising spins of \( \pm 1 \).

Now we consider a spin-3/2 Ising model with pair interactions and a uniaxial potential and with up-down symmetry, which is described by the following Hamiltonian:
\[ -\beta H(S) = \sum_{(ij)} \left[ J_{ij} S_i S_j + K_{ij} S_i^2 S_j^2 + L_{ij} S_i^3 S_j^3 + M_{ij} (S_i^3 S_j^3 + S_j^3 S_i^3) + A_{ij} (S_i^2 + S_j^2) \right], \] (11)
where we assume that
\[ A_{ij} = -\frac{5}{9} K_{ij} \] (12)
and
\[ M_{ij} = -\frac{9}{20} J_{ij} - \frac{91}{180} L_{ij}. \] (13)
Here \((ij)\) in Eq. (11) indicates a summation over pairs of sites. According to Eqs. (7) and (8), we have the following expression up to a constant:

\[-\beta H(\sigma, u) = \sum_{(ij)} \left[ K^2_{ij} \sigma_i \sigma_j + K^2_{ij} u_i u_j + K^2_{ij} \sigma_i u_j \sigma_j \right], \tag{14}\]

where

\[K^2_{ij} = \frac{2}{15} J \omega - \frac{98}{1215} L \omega, \tag{15}\]

\[K^2_{ij} = \frac{16}{81} K \omega \tag{16}\]

and

\[K^1 \omega = -\frac{1}{30} J \omega \omega - \frac{169}{2430} L \omega. \tag{17}\]

In this way, the spin-3/2 Ising model given by Eq. (11) with conditions (12) and (13) is equivalent to an Ashkin-Teller model\(^{19}\) given by Eq. (14). A decoupling relation is given by

\[L \omega = \frac{81}{169} J \omega. \tag{18}\]

Namely it turns out that the spin 3/2-Ising model (11) with conditions (12) and (13) consists of independent two Ising models with spin ±1 when Eq. (18) is satisfied.

To specify the system on a square lattice, we consider the following Hamiltonian:

\[-\beta H(S) = \sum_{r} \left[ J^x S_r S_{r+\hat{x}} + K^x S_r^2 S_{r+\hat{x}}^2 + L^x S_r^3 S_{r+\hat{x}}^3 + M^x S_r S_{r+\hat{x}} (S_r^2 + S_{r+\hat{x}}^2) \right.
\qquad + \Delta^x (S_r^2 + S_{r+\hat{x}}^2) + J^r S_r S_{r+\hat{r}}^+ + K^r S_r^2 S_{r+\hat{r}}^+ + L^r S_r^3 S_{r+\hat{r}}^+ \right.
\qquad \left. + M^r S_r S_{r+\hat{r}}^+ (S_r^2 + S_{r+\hat{r}}^2) + \Delta^r (S_r^2 + S_{r+\hat{r}}^2) \right] \tag{19}\]

with

\[K^t = \frac{27}{40} J^t - \frac{49}{120} L^t, \tag{20}\]

\[M^t = -\frac{9}{20} J^t - \frac{91}{180} L^t, \quad \Delta^t = -\frac{5}{9} K^t \tag{21}\]

for \(t = x, r\). Here \(r = (i, j)\) labels a lattice site and \(\hat{x} = (1, 0)\) and \(\hat{r} = (0, 1)\) are unit vectors in the \(x\) and \(r\) directions, respectively. This is a spin-3/2 Ising model with four interaction parameters, for example, \(J^x, J^r, L^x\) and \(L^r\). In the following, we assume that \(J^x > 0\) and \(J^r > 0\). Its corresponding Ashkin-Teller Hamiltonian is given, up to constant, as follows:

\[-\beta H(\sigma, u) = \sum_{r} \left[ K^x_r (\sigma_{r} \sigma_{r+\hat{x}} + u_r u_{r+\hat{x}}) + K^r_s \sigma_{r} \sigma_{r+\hat{r}} + u_{r+\hat{r}} u_{r} \right.
\qquad \left. + K^x_\tau (\sigma_{r} \sigma_{r+\hat{r}} + u_r u_{r+\hat{r}}) + K^r_\tau \sigma_{r} \sigma_{r+\hat{r}} u_{r+\hat{r}} \right], \tag{22}\]
where we have from Eqs. (15)~(17)

\[ K_2^t = \frac{2}{15}J^t - \frac{98}{1215}L^t, \quad (23) \]

\[ K_4^t = -\frac{1}{30}J^t + \frac{169}{2430}L^t \quad (24) \]

for \( t=x \) and \( \tau \).

For the system (19) with

\[ K^t = L^t = \frac{81}{169}J^t, \quad M^t = -\frac{9}{13}J^t, \quad \Delta^t = -\frac{45}{169}J^t \quad (25) \]

for \( t=x \) and \( \tau \), we have

\[ K_2^t = \frac{16}{169}J^t, \quad K_4^t = 0 \quad (26) \]

for \( t=x \) and \( \tau \) in Eq. (22). Thus the system (19) with conditions (25) is solved in terms of the Onsager solution.

For the system with \( J^r = J^f = J \) and \( L^r = L^f = L \) in Eq. (19), we have \( K_2^r = K_2^f = K_2 \) and \( K_4^r = K_4^f = K_4 \) in Eqs. (22)~(24). A phase boundary of the Ashkin-Teller model, \( K_2 = K_2(\lambda) \), in the \( \lambda - K_2 \) plane, where \( \lambda = K_4/K_2 \), is expressed in the \( J - (L/J) \) plane as follows:

\[ J = \frac{1215}{162 - 98x} K_2 \left( \frac{81 - 169x}{196x - 324} \right), \]

\[ x = \frac{L}{J} \quad (27) \]

A schematic phase diagram is given in Fig. 1 for \( J > 0 \) from the results by Fan and Wu,\(^{13}\) by Knops,\(^{14}\) by Ditzian et al.\(^{15}\) and by Kohmoto et al.\(^{16}\) A paramagnetic region is denoted by I, a partially ordered region by II and a fully ordered region (Baxter phase) by III. Here we assumed that, for a large \( K_4/K_2 \), an asymptotic form of the boundary between regions I and II in the Ashkin-Teller model is proportional to \( (K_2/K_4)^l \) with \( l=1 \). (The boundary between I and II goes to infinity at \( L/J = 81/49 \) if \( l < 1 \) and goes to zero there if \( l > 1 \).) The boundary between II and III goes to infinity at \( L/J = 81/49 \). We notice the point \( L/J = 81/49 \) corresponds to \( K_4/K_2 = \infty \). When \( K = 81J/365, L = 81J/73, M = -369J/365 \) and \( \Delta = -9J/75 \), we have \( K_2 \)
\( K_s = 16J/365 \) and the system is solved in terms of four-state Potts solution. The decoupling point is denoted by \( d \) and the four-state Potts point by \( p \) in Fig. 1.

For the system given by Eq.(19) with anisotropic interactions, it is possible to discuss the critical properties from those of its corresponding Ashkin-Teller model given by Eq. (22). By taking the same limit in Eq. (22) as that done by Kohmoto et al.,\(^\text{16}\) we could find critical exponents as functions of \( \lambda \) and hence functions of \( L^x/J^x \). We give the critical exponent \( \alpha \) in Fig. 2 as an example. A solid line shows predicted values from the extended scaling relations. The decoupling point is denoted by \( d \) and the four-state Potts point by \( p \) in Fig. 2. At \( L^x/J^x = -(18063 + 19440\sqrt{2})/9353 \), \( \alpha \) becomes minus infinity. A dashed line shows an Ising value and exists between \( 81/73 < L^x/J^x < 81/49 \). The point \( L^x/J^x = 81/49 \) corresponds to \( K_s/K_t = \infty \).

In short, we have shown that a spin-3/2 Ising model with up-down symmetry is expressed in terms of an Ashkin-Teller model. Critical properties for spin-3/2 Ising models with other interaction parameters including competing interactions on the square lattice and on the other lattices might have more rich critical phenomena and will be discussed in the future.

---