Deformation of Integrable Systems
Associated with Symmetric Space of the Lie Algebra

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Integrable deformation of nonlinear partial differential equations is obtained by the space and
time variation of the spectral parameter associated with the original Lax pair. Equations associated
with the symmetric spaces of a general Lie algebra are analysed. A new class of integrable equations
is generated of which cylindrical NLS equation, space-time dependent wave-wave interaction equa-
tions turn out to be special cases. Furthermore, corresponding to third order flow, a generalisation
of the KdV equation is also obtained.

§ 1. Introduction

Integrable nonlinear partial differential equations are usually generated through
the use of an assumed form of a Lax pair belonging either to a classical or affine Lie
algebra. A very interesting and powerful method was suggested by Fordy and
Kulish. Here these authors generated a new class of nonlinear Schrödinger equation by taking recourse to a symmetric space decomposition of Lie algebra. Here in
this communication we show that a more general class of nonlinear integrable
systems can be constructed if we consider the eigenvalue parameter ‘λ’ to be space
time dependent. It is interesting to note that the space time variation of the spectral
parameter imposes some kind of restriction on the type of the nonlinear field equation.
In general we pursue our discussion with a general Lie algebra, which yields some
generalisation of the cylindrical NLS equation, wave-wave interaction system, and
cylindrical KdV equations are obtained. These equations are called deformed inte-
grable systems.

§ 2. Formulation

Suppose ‘g’ is a simple Lie algebra which in the Cartan-Weyl basis has the
following commutation relations:

\[ [h_i, h_j] = 0 \forall h_i, h_j \in \mathfrak{h}, \]

\[ [h, e_a] = \alpha(h)e_a, \forall h \in \mathfrak{h}, a \in \phi, \]

\[ [e_a, e_{-a}] = h_a = \sum_i d_{ai} h_{ai}, \]

\[ [e_a, e_{\beta}] = \begin{cases} N_{a\beta} e_{a+\beta} & \text{if } a + \beta \neq 0, \\ 0 & \text{if } a + \beta \in \phi, \end{cases} \]

where \( \mathfrak{h} \) is the Cartan subalgebra, \( \phi \) denoting the root space.
A homogeneous space of Lie group $G$ is any differentiable manifold $M$ on which
$G$ acts transitively. A subgroup of $G$ which leaves a given point $P_0 \in M$ fixed is called
the isotropy group at $P_0$ and is defined by

$$ K = \{ g \in G ; \ g \cdot P_0 = P_0 \} . $$

By a standard result each such $M$ can be identified as a coset space $G/K$. Thus if $g$
and $k$ be the Lie algebra corresponding to the groups $G$ and $K$, then

$$ g = k \oplus m ; \quad [k, k] \subset k . $$
$$ [k, m] \subset m ; \quad [m, m] \subset k . $$

From the adjoint representation we can define the metric, curvature and torsion, as

$$ g(X, Y) = \text{tr}(adX adY) , $$
$$ g_{ij} = g(X_i, X_j) , $$
$$ R(X_k, X_i) X_j = R_{jki} X_i , $$

where

$$ (R(x, y) z)_{P_0} = -[[x, y], z] , $$
$$ T(x_j, X_k) = T_{jk} X_i . $$

For space of constant curvature

$$ R'_{jki} = K(\delta_k^i g_{j\lambda} - \delta_i^j g_{j\lambda}) , $$

where $K$ is the constant Gaussian curvature.

Integrable systems associated with symmetric algebra were deduced by Fordy
and Kulish by starting from a Lax pair

$$ \phi_x = (\lambda E + Q(r, t)) \phi , $$
$$ = L(\lambda) \phi , $$
$$ \phi_t = P(x, t, \lambda) \phi $$

by involving several Lie algebraic characters for $Q$ and $P$. Here in our formulation
we also assume that the spectral parameter is no more a constant but varies with $x$
and $t$ according to

$$ \frac{\partial \lambda}{\partial x} = l(x, t, \lambda) , $$
$$ \frac{\partial \lambda}{\partial t} = p(x, t, \lambda) , $$

where $l, p$ are functions of $x, t$ and $\lambda$. Therefore if we now demand consistency, then
we get

$$ \lambda_t E + Q_t + (\lambda E + Q) P = P_x + P(\lambda E + Q) $$
or \( P_x - Q_t = E \lambda_t + [\lambda E + Q, P] \). \( (3) \)

Also consistency should be imposed on (2) leading to
\[ l_t = p_x. \] \( (4) \)

To proceed further we need to specify the nature of \( Q \) and \((l, p)\) functions.

**Case I**

In general \( E \in \mathfrak{h} \) and \( Q \subset m, P \subset g \). Therefore we set \( P = P_h + P_m \), also Eq. (3) breaks up into
\[ Q_t = P_{mx} - [Q, P_h] - \lambda[E, P_m], \] \( (5) \)
\[ P_{xx} = [Q, P_m] + \lambda E. \] \( (6) \)

Let us consider a second order flow pertaining to
\[ P = P_{xx}^2 + P_1 \lambda + P_0. \] \( (7) \)

Also we assume
\[ l(x, t, \lambda) = \bar{b} \lambda, \quad p(x, t, \lambda) = a \lambda^2, \] \( (8) \)
\( \bar{b}, a, \) being some functions of \((x, t)\).

From Eq. (4) we have
\[ a_x = -\bar{b} a; \quad \bar{b}_t = 0. \] \( (9) \)

Since \( Q \subset m \), we can represent \( Q \) as
\[ Q = \sum_{\alpha \in \theta^+} (q^\alpha e_\alpha + k^\alpha e_{-\alpha}), \] \( (10) \)

where \( \theta^+ \) is the subset of the positive roots for which \( a(E) \neq 0 \).

\( e_\alpha, e_{-\alpha} \) are the generators introduced previously. Substituting (7) and (10) in (5) and (6), we equate various powers of \( \lambda \) and also coefficients of \( e_\alpha \) and \( e_{-\alpha} \), whence we get
\[ P_{0m} = \sum_{\alpha \in \theta^+} \frac{1}{a(E)} [(q_x^\alpha + \bar{b} q)e_\alpha - (k_x^\alpha + \bar{b} k^\alpha)e_{-\alpha}], \] \( (11) \)
\[ P_{0h} = \partial^{-1}[Q, P_{0m}] \]
\[ = \sum_{\alpha, \beta \in \theta^+} \frac{1}{a(E)} (q^\alpha k^\beta + 2\partial^{-1} \bar{b} q^\alpha k^\beta)[e_\alpha, e_{-\beta}]. \] \( (12) \)

Therefore the required nonlinear equations are
\[ Q_t = P_{0mx} - [Q, P_{0h}], \] \( (13) \)

which immediately entails
\[ a(E) \cdot q_t^\alpha = q_x^{\alpha x} + q_x^\alpha \bar{b} + q^\alpha \bar{b}_x \]
\[ + \sum_{\beta, \tau, \delta \in \theta^+} R_{\beta, \tau, \delta}^\alpha \{ q^\beta q^\tau k^\delta + 2\partial^{-1} q^\beta q^\tau k^\delta \bar{b} \}, \] \( (14) \)
\[-\alpha(E)\dot{k}^a = k_x^a \dot{x} + \dot{k}_x^a \vec{b} + \dot{k}^a \vec{b}_x \]
\[+ \sum_{\beta, \gamma, \delta \in \theta^\ast} R_{\beta, \gamma, \delta}^{-\beta, \gamma, \delta} \left[ k^\beta q^\delta k^\gamma + 2k^\beta \partial^{-1} k^\gamma q^\beta \vec{b} \right] \ldots. \tag{15}\]

For SU(4) algebra of rank \(l = 3\) there are the set of \(\frac{1}{2} l(l + 1) = 6\) positive roots with basic elements,

\[X_1 = e_{12}, \quad X_2 = e_{13}, \quad X_3 = e_{14}, \quad X_4 = e_{23}, \]
\[X_5 = e_{24}, \quad X_6 = e_{34}, \]

where

\[(e_{pq})_{jk} = \delta_{jp} \delta_{kq}. \quad \text{for } p, q = 1, 2, 3, 4; p \neq q\]

From

\[g(e_{pq}, e_{rs}) = 8 \delta_{qr} \delta_{ps}, \quad p, q, r, s = 1, 2, 3, 4; p \neq q, \ r \neq s\]

the non-zero Killing forms are

\[g_{i,-i} = g_{-i,i} = 8; \quad i = 1, 2, 3, 4, 5, 6.\]

Thus the Riemann curvature tensors which have a non-zero value are

\[R_{ij} = K_{ij}, \quad R_{jk} = -K_{jk}, \quad i, j, k = 1, 2, 3, 4, 5, 6.\]

Thus for SU(4) equations (14) and (15) turn out to be

\[\alpha(E)q_t^a = q_x^a \dot{x} + q_x^a \vec{b} + q^a \vec{b}_x \]
\[+ 8K \sum_{\beta = 1}^6 \left[ (q^\beta q^\alpha k^\beta - q^\beta q^\beta k^\alpha) + 2q^\beta \partial^{-1} (q^\alpha k^\beta - q^\alpha k^\beta) \vec{b} \right], \]
\[-\alpha(E)k_t^a = k_x^a \dot{x} + \dot{k}_x^a \vec{b} + \dot{k}^a \vec{b}_x \]
\[+ 8K \sum_{\beta = 1}^6 \left[ k^\beta q^\alpha k^\beta + 2k^\beta \partial^{-1} q^\alpha \vec{b} k^\gamma \right], \]

where \(\alpha = 1, 2, 3, 4, 5, 6.\) Furthermore it is interesting to observe that from the equation involving \(p_{2k}\) and \(p_{2m}\) we obtain another constraint on the functions \(a, \vec{b},\) it is given as

\[a_x + a^2 \frac{2}{2} = 0,\]

whence we have \(a = 2/x, \ \vec{b} = 1/x.\)

Thus Eqs. (14) and (15) with the form of \(\vec{b}\) given above yields a pair of generalised NLS equation with non-local term and cylindrical in character.

Case II

Another situation, where we have obtained interesting generalisation of integrable system is given by

\[\lambda_x = \vec{c},\]
\[ \lambda_t = \lambda b, \quad (16) \]

\( \tilde{c}, b \) being arbitrary functions of \((x, t)\). Here also for a second order flow we get

\[ P_{0m} = \sum_{a \in \theta^*} \frac{1}{a(E)} \left[ q_a^s + x(b - 2 \tilde{c})q_a^a \right] e_a \]

\[ - \sum_{a \in \theta^*} \frac{1}{a(E)} \left[ k_a^s - x(b - 2 \tilde{c})k_a^a \right] e_{-a} \quad (17) \]

and

\[ P_{0h} = \sum_{a \in \theta^*} -\frac{1}{a(E)} q_a^s \cdot k^a \left[ e_a, e_{-a} \right] - \tilde{c} (b - 2 \tilde{c}) \frac{x^2}{2} E. \quad (18) \]

Now Eq. (2) due to consistency and along with \(b = 2 \tilde{c}\) yields

\[ \tilde{c} = -\frac{1}{2t}; \quad b = -\frac{1}{t} \]

and the equation sets are

\[ q_t^a = \frac{1}{a(E)} q_x^a + \tilde{c} q^a + \frac{1}{a(E)} \sum_{\beta, \tau \in \theta^*} R_{\beta, \tau}^{a, -a} q^\beta q^\tau k^a, \]

\[ -k_t^a = -\frac{1}{a(E)} k_x^a - \tilde{c} k^a - \frac{1}{a(E)} \sum_{\beta, \tau \in \theta^*} R_{-\beta, \tau}^{a, -a} k^\beta k^\tau. \quad (19) \]

Equation (19) yields another generalisation of the dispersive wave equation with explicit time dependence and local in nature.

**Case III**

In this last case we consider a third order flow along with the condition

\[ \lambda_t = a \lambda^3; \quad \lambda_x = 0. \]

Therefore in this case we assume

\[ P = P_0 \lambda^3 + P_2 \lambda^2 + P_1 \lambda + P_0 \]

and substituting in (5) and (6) we proceed as before and obtain

\[ P_{3m} = 0, \]

\[ P_{3h} = ax E, \]

\[ P_{2m} = \sum_{a \in \theta^*} \frac{1}{a(E)} \left[ (a + a(E)A)q_a^x + axq_a^s \right] e_a - \left[ (a - a(E)A)k_a^x + axk_a^s \right] e_{-a}, \]

\[ P_{2h} = \sum_{a, \beta \in \theta^*} -\frac{a}{a(E)} \left( xq_a^xk_a^x + \partial^{-1} q_a^xk^s \right) \left[ e_a, e_{-a} \right], \quad (20) \]

finally we get

\[ a^2(E) g_t^a = axg_{xx}^a + \left( 3a + a(E)A \right) g_{xx}^a \]
\[ + \sum_{\delta, \sigma, \eta, \xi, \zeta, \xi, \eta, \zeta} \{aR_{\delta \eta - \xi}^\eta q \xi \{x q^\xi k^\eta + \partial^{-1} q^\xi k^\eta \} \\
+ q^\delta \{2 q^\xi k^\eta + x(q^\xi k^\eta)_x\} - q^\delta R_{\delta \eta - \xi}^\eta \\
\times \partial^{-1}[(2a + \alpha(E)A)(k^\eta q^\xi)_x + ax(q^\xi k^\eta_x - q^\eta_k^\xi) \\
- a(xq^\xi k^\eta + \partial^{-1} q^\xi k^\eta)(R_{\delta \eta - \xi}^\xi k^\eta q^\eta + R_{\eta \delta - \xi}^\xi q^\xi k^\eta)] \] (21)

and

\[ a^2(E)k^\xi_t = axk^\xi_{xxx} + (3a - \alpha(E)A)k^\xi_x \\
- \sum_{\delta, \sigma, \eta, \xi, \zeta, \xi, \eta, \zeta} \{aR_{\delta \eta - \xi}^\eta k \xi \{x q^\xi k^\eta + \partial^{-1} q^\xi k^\eta \} \\
+ k^\delta \{2 q^\xi k^\eta + x(q^\xi k^\eta)_x\} - k^\delta R_{\delta \eta - \xi}^\eta \\
\times \partial^{-1}[(2a + \alpha(E)A)(k^\eta q^\xi)_x + ax(q^\xi k^\eta_x - q^\eta_k^\xi) \\
- a(xq^\xi k^\eta + \partial^{-1} q^\xi k^\eta)(R_{\delta \eta - \xi}^\xi k^\eta q^\eta + R_{\eta \delta - \xi}^\xi q^\xi k^\eta)] \]. (22)

Equations (21) and (22) give a generalisation of KdV class of equations.

\section{3. Conserved quantities}

Since one of the most important criterion for complete integrability is the existence of infinite number of conserved quantities, in this section, we proceed to demonstrate the same for our deformed equations. However there are two aspects which make the proof difficult, 1) one is the coupled nature of the equations due to the symmetric Lie algebra, 2) the other is the explicit space-time dependence. Therefore here we take a simple example in the spirit of the above discussion.

Consider the Lax pair,

\[ \phi_{xx} + (\lambda + u) \phi = 0, \]
\[ \phi_t = A\phi_x + B\phi, \] (23)

along with

\[ \lambda_x = -\frac{1}{12t}; \quad \lambda_t = -\frac{\lambda}{t} \]

which is nothing but a scalar variant of Eqs. (1) and (2), whence we can generate the cylindrical KdV equation

\[ u_t + 6uu_x + u_{xxx} + \frac{u}{2t} = 0. \] (24)

We can now rewrite this equation as

\[ u_t + K(u) = 0. \] (25)

The linearized form of this is given by

\[ \eta_t + K'(u)\eta = 0, \] (26)
where \( K'(u) \) is the Fréchet derivative of \( K(u) \). This form of the simplest deformed KdV equation was known long ago. To prove the existence of conserved quantities, we consider the adjoint of (26).

\[
\sigma + K'(u)\sigma = 0.
\]  

(27)

The \( u=0 \) part of (27) is solved by

\[
\sigma = t^{1/3} \exp(k^3 t - kx),
\]

so we search for the solution of (27) as

\[
\sigma = t^{1/3} \exp\left(k^3 t - kx + \int_{-a}^{x} Tdx\right)
\]

with

\[
T = \sum_{n=0}^{\infty} k^{-n} T_n.
\]

Substituting in Eq. (27) and equating various powers of \( k^{-n} \), we obtain the following recursion relation:

\[
\int_{-a}^{x} T_n dx - 6u \delta n_{+1,0} + 6u T_n + T_{n, xx} - 3T_{n+1, x}
\]

\[
+ 3 \sum_{i=0}^{n} T_i T_{n-i, x} + 3 T_{n+2} - 3 \sum_{i=0}^{n+1} T_i T_{n+i - i}
\]

\[
+ \sum_{i=0}^{n} \sum_{j=0}^{n-1} T_i T_j T_{n-i-j} = 0,
\]

(28)

which immediately yields

\[
T_0 = 0,
\]

\[
T_1 = 2u,
\]

\[
T_2 = 2ux,
\]

\[
T_3 = 2u^2 + 2u_{xx} + \frac{1}{3t} D^{-1} u,
\]

\[
T_4 = 2u_{xxx} + 3uu_x + \frac{2u}{3t}
\]

(29)

and so on. In case of equations not containing explicit \( (x, t) \) dependence these would be the candidates for conserved densities, but in the present case the situation is different. Consider for example,

\[
\frac{d}{dt}\left(\int_{-\infty}^{x} T_1 dx\right) = -6u^2 - 2u_{xx} - \frac{1}{2t}\left(\int_{-\infty}^{x} T_1 dx\right).
\]

(30)

Now let us take limit \( x \to \infty \) with the supposition that \( u \to 0 \) as \( x \to \infty \), whence we get

\[
\frac{d}{dt}\left(\int_{-\infty}^{\infty} T_1 dx\right) = -\frac{1}{2t}\left(\int_{-\infty}^{\infty} T_1 dx\right)
\]

(31)
from which we have
\[
C_1 = t^{3/2} \left( \int_{-\infty}^{\infty} T_1 \, dx \right)
\]
which is a conserved density. Also by a laborious computation we can show
\[
C_2 = 0,
\]
\[
C_3 = t^{3/2} \left( \int_{-\infty}^{\infty} T_3 \, dx \right),
\]
\[
C_4 = t^{3/2} \left( \int_{-\infty}^{\infty} T_4 \, dx \right)
\]
which are some of the other conserved densities. However at each stage a computation is necessary to get the exact form of \( C_i \) from that of \( T_i \). The same treatment also holds for the DeNLS equation (deformed NLS equation). We start from the following set: (again we ignore the Lie algebraic structure, leading to the multicomponent situation, in which case the computation becomes terribly complicated)
\[
q_t = \frac{1}{a} q_{xx} - \frac{1}{2t} q + \frac{1}{a} (q)^2 k,
\]
\[
k_t = -\frac{1}{a} k_{xx} - \frac{1}{2t} k - \frac{1}{a} (q)^2.
\]
If we perform the linearization we get \((q \rightarrow q + \varepsilon Q, k \rightarrow k + \varepsilon K)\)
\[
\begin{pmatrix}
Q \\
K
\end{pmatrix}_t = \begin{pmatrix}
\alpha^{-1} q_{xx} - \frac{1}{2t} + 2\alpha^{-1} qk & \alpha^{-1} q^2 \\
-\alpha^{-1} k^2 & -\alpha^{-1} q_{xx} - \frac{1}{2t} - 2\alpha^{-1} qk
\end{pmatrix}
\begin{pmatrix}
Q \\
K
\end{pmatrix} = M \begin{pmatrix}
Q \\
K
\end{pmatrix}.
\]
Now as before adjoint of (35) is
\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix}_t = M_{\Lambda a} \begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix},
\]
where \( q = k = 0 \). A solution of (36) is seen to be
\[
\sigma_2 = \sqrt{t} \exp (\nu^2 t - \sqrt{a} \nu x),
\]
whence we get
\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2
\end{pmatrix} = \begin{pmatrix}
1 \\
b(x, t)
\end{pmatrix} \sqrt{t} \exp \left( \left( \nu^2 t - \sqrt{a} \nu x + \int_{-\infty}^{x} T \, dx \right) \right).
\]
By substituting in Eq. (36) we obtain the following two equations for \( b \) and \( T \):
\[
\begin{align*}
b_t + 2bR^2 + \int_{-\infty}^{x} T \, dx + \alpha^{-1} (b_{xx} + 2Tb_x + bT_x + bT^2 - 2\sqrt{a} \nu b_x - 2\sqrt{a} \nu bT + 2qkb - k^2) &= 0,
\end{align*}
\]
\[ a \int_{-a}^{x} T_x dx - T_x^2 - T^2 + 2a^{1/2} r T - 2qK + qa^2 b = 0. \] (39)

If we now substitute
\[ T = \sum_{n=1}^{\infty} T_n r^{-n}; \quad b = \sum_{m=2}^{\infty} b_m r^{-m}, \] (40)
we can determine \((T_n, b_m)\) recursively for example,
\[ T_1 = a^{-1/2} q k, \]
\[ T_2 = a^{-1} q k_x + \frac{1}{2t} D^{-1}(qk), \]
\[ b_2 = \frac{1}{2a} k^2, \]
\[ b_3 = \frac{1}{2a^{3/2}} (k^2)_x, \]
\[ T_3 = \frac{1}{a^{3/2}} (qK_{xx} + q^2 K^2) \frac{1}{a^{1/2} t} D^{-1}(qK_x) + \frac{a}{2t^2} D^{-2}(qK), \] (41)

and so on. Lastly a computation similar to above yields
\[ \frac{d}{dt} \left( \int_{-a}^{x} T_1 dx \right) = \frac{1}{t} \left( \int_{-a}^{x} T_1 dx \right). \] (42)

Therefore \(G_1 = t (\int_{-a}^{x} T_1 dx)\) is conserved and \((d/dt)(\int_{-a}^{\infty} T_2 dx) = -(2/t)(\int_{-a}^{\infty} T_2 dx).\)

Then
\[ C_2 = t^2 \left( \int_{-a}^{\infty} T_2 dx \right) \] (43)

is another.

\section*{§ 4. Discussion}

In our above analysis we have shown that by the assumption of space-time variation of the spectral parameter in the zero curvature representation, a class of generalised integrable nonlinear equations can be constructed, some of which are actually realized in nature. Such as the \((x, t)\) dependent KdV can occur in a plasma (ion-acoustic wave) with a density gradient. Also the cylindrical NLS equation is connected to the cylindrical Heisenberg spin chain which is supposed to occur in a magnetic system with a symmetry axis.

Usually one always refers to a nonlinear equation as an integrable one if it can be associated with a pair of linear equations — the Lax pair, because then it becomes possible to utilise the machinery of inverse scattering transform to obtain the solutions. Also one can write down using the properties of Riccati equations, the infinite number of conservation laws. In the present situation since we are starting from a Lax pair (Eqs. (1) and (2)) all these are guaranteed from the very beginning.
Lastly we can comment on the relevance of such equations in actual physical problems. For a long time it has been observed that when a nonlinear equation describes the propagation of a wave in an inhomogeneous medium the equation becomes explicitly dependent on space or time or both. In general Lax pair associated with such equations do have spectral parameter varying with either space or time. As an important example we can quote the example of nonlinear Schrödinger equation with linear or parabolic density profiles, already studied in the literature, whose generalisation is Eq. (15a) or (19). It may be mentioned further that equation of the type (15a) will occur if one considers the propagation of a polarised optical wave in a nonlinear plasma with inhomogeneous density.

There exists another situation where equation of this type occurs. In this case one considers the wave propagation in a bounded non-linear medium. An important example is that of cylindrical KdV equation with explicit dependence on time. Our Eq. (21) is actually a generalisation of such system, when there are more than one propagating wave interacting with each other, within an inhomogeneous medium with density gradient.

Lastly we have tested the existence of infinite number of conservation laws in a scalar version of our type of equation and also in a simplified form of deNLS equation. In each case it is possible to construct the conserved quantities though we have not been able to prove their existence in general.

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